Exercise 1A #9

Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Prove that if $c, d \in \mathbb{R}$ and $a \le c < d \le b$, then f is Riemann integrable on $[c,d]$.

By Exercise 1A#3, $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $P = \{x_0, ..., x_n\}$ where $x_0 = a$ and $x_n = b$,

$$U(f,P) - L(f,P) < \varepsilon.$$

Refine the partition to be $P' = P \cup \{c,d\}$ to guarantee that our investigated partition includes c, d.
By Theorem 1.5,

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P),$$

and since $U(f,P) - L(f,P) < \varepsilon$

$$U(f,P') - L(f,P') < \varepsilon.$$

We now construct a partiton of $[c,d]$ with $P^* = P' \cap [c,d]$. We have

$$L(f,P^*) \le \sup_{P^*} L(f,P^*) \le L(f,P')$$

and

$$\inf_{P^*} U(f,P^*) \le U(f,P^*) \le U(f,P').$$

By multiplying the first inequality by -1 and adding them together, we get

$$\begin{aligned}\inf_{P^*} U(f,P^*) - \sup_{P^*} L(f,P^*) &\le U(f,P') - L(f,P') \\ &< \varepsilon.\end{aligned}$$

For an arbitrary $\varepsilon > 0$, we have constructed a partition of $[c,d]$ such that the difference of their upper and lower Riemann sums was less than ε. Thus, f is Riemann integrable on $[c,d]$.