MTH 650

Presentation 1 - Section 1B # 2

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Exercise 2

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is bounded and let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$. We want to show that f is Riemann integrable if and only if $L(-f,[a,b]) = -L(f,[a,b])$.
Before showing both directions, consider the following. For any nonempty subinterval $I \subseteq [a,b]$,

$$\inf_I(-f) = -\sup_I f$$

and

$$\sup_I(-f) = -\inf_I(f)$$

because for any set $A \subseteq \mathbb{R}$ we know $\inf(-A) = -\sup A$ and $\sup(-A) = -\inf A$. Let $I_j = [x_{j-1}, x_j]$. By definition for every partition P of $[a,b]$,

$$\begin{aligned}
L(-f,P,[a,b]) &= \sum_{j=1}^{n}(x_j - x_{j-1}) \inf_{I_j}(-f) \\
&= \sum_{j=1}^{n}(x_j - x_{j-1})(-\sup_{I_j} f) \\
&= -U(f,P,[a,b]).
\end{aligned}$$

Taking the supremum and infimum over all partitions gives us

$$\begin{aligned}
\sup_P L(-f,P,[a,b]) &= \sup_P(-U(f,P,[a,b])) \\
L(-f,[a,b]) &= -\inf_P(U(f,P,[a,b]) \\
L(-f,[a,b]) &= -U(f,[a,b]). \qquad (1)
\end{aligned}$$

To show the forward direction suppose f is Riemann integrable. Then

$$L(f,[a,b]) = U(f,[a,b]).$$

Using equation (1) we can negate both sides to give us

$$-L(-f,[a,b]) = U(f,[a,b]).$$

By transitivity we get

$$-L(-f,[a,b]) = L(f,[a,b]).$$

Negating both sides again gives

$$L(-f,[a,b]) = -L(f,[a,b]).$$

To show the backward direction suppose $L(-f,[a,b]) = -L(f,[a,b])$. Replacing the left hand side with $-U(f,[a,b])$ from equation (1) gives us

$$-U(f,[a,b]) = -L(f,[a,b]).$$

Negating both sides gives

$$U(f,[a,b]) = L(f,[a,b]).$$

which means f is Riemann integrable.
Therefore f is Riemann integrable if and only if $L(-f,[a,b]) = -L(f,[a,b])$. $\square$