Excercises 1A, Problem 6

Suppose that $f : [a, b] \to \mathbf{R}$ is Riemann integrable. Suppose $g : [a, b] \to \mathbf{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a, b]$. Prove that g is Riemann integrable and that

$$\int_a^b g = \int_a^b f$$

Proof: We have a function $g(x)$ that is the same as the Riemann integrable function $f(x)$ except for a finite number of points on the interval $[a, b]$. Intuitively, the integral for these functions should be the same since the integral of these finite points is zero since they have length zero. Therefore, the contribution of these intervals to the Riemann sum will be zero.

First, we consider the case where there is only one difference between $f(x)$ and $g(x)$, call this point $c \in [a, b]$. Now, let a new function $h = f - g$, which has only one point equal to c and is zero everywhere else. To prove that h is Riemann integrable, let $\varepsilon > 0$ and less than the minimum distance between $[a, c]$ and $[b, c]$. Let $w = \frac{\varepsilon}{4|h(c)|}$, where $|h(c)|$ is the absolute value of h at point c. Let there be a partition $P = \{a, c - w, c + w, b\}$. Let $M = \sup h$ and $m = \inf h$. We use the definition of upper and lower Riemann sums,

$$U(h, P, [a, b]) - L(h, P, [a, b]) = \sum_k (x_k - x_{k-1})\big(M_{I_k} - m_{I_k}\big),$$

to get:

$$\begin{aligned} U(h, P, [a, b]) - L(h, P, [a, b]) = {} & (c - w - a)\Big(M_{[a,c-w]} - m_{[a,c-w])}\Big) \\ & + (c + w - c + w)\Big(M_{[c-w,c+w]} - m_{[c-w,c+w]}\Big) \\ & + (b - c + w)\Big(M_{[c+w,b]} - m_{[c+w,b]}\Big) \end{aligned}$$

Here, only the second term has nonzero value, since the supremum (M) minus the infemum (m) of the intervals within the partitions $[a, c - w]$ and $[c + w, b]$ are equal to zero, since all values outside of c are zero given the construction of our function h. Thus,

$$\begin{aligned} U(h, P, [a, b]) - L(h, P, [a, b]) = & 0 + (c + w - c + w)\Big(M_{[c-w,c+w]} - m_{[c-w,c+w]}\Big) + 0 \\ = & 2w|h| \\ = & 2|h|\frac{\varepsilon}{4|h|} \\ = & \frac{\varepsilon}{2} \\ < & \varepsilon \end{aligned}$$

The length between $c+w, c-w$ is $2w$, and the sup − inf over this interval is equal to the absolute value of $|h|$. Therefore, $U(f,P,[a,b]) - L(f,P,[a,b]) < \varepsilon$, and so h is Riemann integrable.

Now we know f and h are Riemann integrable. Since $h = f - g$, we can rewrite as $g = f - h$. Assume, without loss of generality, that $h(c) > 0$. From previous theorems (Exercises 1A.4, 1A.5), we can say that

$$\int_a^b f - h = \int_a^b f - \int_a^b h$$
$$\int_a^b g = \int_a^b f - \int_a^b h$$

We have proven the case where g is integrable where the point of difference c falls between (a,b). What if $a = c$, or $c = b$? We just need to adjust our partitions. For the case where $a = c$, let $P = \{a, a + \frac{\varepsilon}{2|h|}, b\}$. Then we have:

$$\begin{aligned}
U(h,P,[a,b]) - L(h,P,[a,b]) =& (a + \frac{\varepsilon}{2|h|} - a)\Big(M_{[a,a+\frac{\varepsilon}{2|h|}]} - m_{[a,a+\frac{\varepsilon}{2|h|}])}\Big) \\
&+ (b - a + \frac{\varepsilon}{2|h|})\Big(M_{[a+\frac{\varepsilon}{2|h|},b]} - m_{[a+\frac{\varepsilon}{2|h|},b])}\Big) \\
=& \frac{\varepsilon}{2|h|}|h| + 0 \\
=& \frac{\varepsilon}{2} \\
<& \epsilon
\end{aligned}$$

Similarly for the case where $c = b$, let the partition for this specific case be $P = \{a, b - \frac{\varepsilon}{2|h|}, b\}$. It follows that

$$\begin{aligned}
U(h,P,[a,b]) - L(h,P,[a,b]) =& (b - \frac{\varepsilon}{2|h|} - a)\Big(M_{[a,b-\frac{\varepsilon}{2|h|}]} - m_{[a,b-\frac{\varepsilon}{2|h|}])}\Big) \\
&+ (b - b - \frac{\varepsilon}{2|h|})\Big(M_{[b-\frac{\varepsilon}{2|h|},b]} - m_{[b-\frac{\varepsilon}{2|h|},b])}\Big) \\
=& 0 + \frac{\varepsilon}{2|h|}|h| \\
=& \frac{\varepsilon}{2} \\
<& \epsilon
\end{aligned}$$

Now we must show that since f and h are Riemann integrable, g is Riemann integrable as well. This statement is true, as seen in the exercises Axler 1A.4 and 1A.5.

$$\int_a^b f - h = \int_a^b f - \int_a^b h$$
$$\int_a^b g = \int_a^b f - \int_a^b h$$

Furthermore, we can prove that $\int_a^b f = \int_a^b g$. Take the following definition of a Riemann integrable function on a closed and bounded interval:

$$\sup L(h, P, [a, b]) = \inf U(h, P, [a, b])$$

Since $L(h, P, [a, b]) = 0$, then the upper and lower Riemann sums for the function h must be 0, and thus:

$$\int_a^b g = \int_a^b f - \int_a^b h$$
$$\int_a^b g = \int_a^b f - 0$$
$$\int_a^b g = \int_a^b f$$

We have proven the function g is Riemann integrable when there is one point of difference between f and g. Now we will prove that this holds for any *finite* number, n, of differences with f.

Assume that if g has exactly n points that differ from f in the interval $[a, b]$ then g is Riemann integrable.

Now, Suppose $n \in \mathbf{Z}^+$. Assume g has $n + 1$ differences on $[a, b]$. Then we have the following partition of $[a, b]$.

$$P = a \leq c_1 < ... < c_n < c_{n+1} \leq b$$

Let $p \in (c, c_{n+1})$ be a point strictly between c_n and c_{n+1}. Then g has exactly n points of difference in the interval $[a, p]$ and is thus integrable on $[a, p]$ given our assumption. This also means that there is exactly one point of difference on the interval $[p, b]$, and thus g is also Riemann integrable over the interval $[p, b]$ (which we proved above).

This is true for any $n + 1$, and therefore, g is Riemann integrable on $[a, b]$ (see also result from Exercise 1A.10).

In addition, the Riemann integral of these finite points is zero, as shown above for single points. Therefore, $\int_a^b f = \int_a^b g$ in this case as well. □