REAL VARIABLES PRESENTATION 1

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Step 1. Understanding the Function and Goal

Let $a \leq s < t \leq b$. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } s < x < t, \\ 0 & \text{otherwise.} \end{cases}$$

We aim to prove that f is Riemann integrable on $[a, b]$ and that

$$\int_a^b f = t - s.$$

Step 2. Preliminaries from the Given Definitions and Theorems

- **Partition**: A partition P of $[a, b]$ is a finite list $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- **Lower and Upper Riemann Sums**: For a bounded function f and partition P,

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f,$$

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

- **Inequalities with Riemann Sums**: If P' is a refinement of P, then

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

- **Comparison of Lower and Upper Sums**: For any partitions P and P',

$$L(f, P, [a, b]) \leq U(f, P', [a, b]).$$

- **Lower and Upper Riemann Integrals**:

$$L(f,[a,b]) = \sup_P L(f,P,[a,b]), \quad U(f,[a,b]) = \inf_P U(f,P,[a,b]).$$

- **Integrability Criterion**: f is Riemann integrable if

$$L(f,[a,b]) = U(f,[a,b]),$$

in which case

$$\int_a^b f = L(f,[a,b]) = U(f,[a,b]).$$

Step 3. Showing $L(f,[a,b]) = t-s$ and $U(f,[a,b]) = t-s$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < \delta < \min\left\{\frac{t-s}{2}, \frac{\varepsilon}{2}\right\}.$$

Define a partition $P = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ by

$$x_0 = a, \quad x_1 = s, \quad x_2 = s+\delta, \quad x_3 = t-\delta, \quad x_4 = t, \quad x_5 = b.$$

Now compute the lower and upper sums:

- On $[x_0, x_1] = [a, s]$: $f(x) = 0$ $m_1 = 0$, $M_1 = 0$.
- On $[x_1, x_2] = [s, s+\delta]$: $f(s) = 0$, but $f(x) = 1$ for $x > s$ $m_2 = 0$, $M_2 = 1$.
- On $[x_2, x_3] = [s+\delta, t-\delta]$: $f(x) = 1$ $m_3 = 1$, $M_3 = 1$.
- On $[x_3, x_4] = [t-\delta, t]$: $f(x) = 1$ for $x < t$, but $f(t) = 0$ $m_4 = 0$, $M_4 = 1$.
- On $[x_4, x_5] = [t, b]$: $f(x) = 0$ $m_5 = 0$, $M_5 = 0$.

Then:

$$L(f,P,[a,b]) = 0 + 0 + (t-\delta-(s+\delta)) \cdot 1 + 0 + 0 = t - s - 2\delta,$$
$$U(f,P,[a,b]) = 0 + \delta \cdot 1 + (t-\delta-(s+\delta)) \cdot 1 + \delta \cdot 1 + 0 = t - s.$$

So,

$$U(f,P,[a,b]) - L(f,P,[a,b]) = 2\delta < \varepsilon.$$

By the definition of lower and upper integrals,

$$L(f,P,[a,b]) \le L(f,[a,b]) \le U(f,[a,b]) \le U(f,P,[a,b]),$$

which implies

$$t - s - 2\delta \le L(f,[a,b]) \le U(f,[a,b]) \le t - s.$$

Since δ can be made arbitrarily small, it follows that

$$L(f, [a, b]) \geq t - s \quad \text{and} \quad U(f, [a, b]) \leq t - s.$$

Combining with (1) and (2), we conclude:

$$L(f, [a, b]) = t - s \quad \text{and} \quad U(f, [a, b]) = t - s.$$

Step 4. Conclusion

Since $L(f, [a, b]) = U(f, [a, b]) = t - s$, the function f is Riemann integrable on $[a, b]$, and

$$\int_a^b f = t - s.$$

Final Result

$$\boxed{t - s}$$