Presentation #1

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Problem 1B#3

3 Suppose $f, g\colon [a,b] \to \mathbf{R}$ are bounded functions. Prove that

$$L(f,[a,b]) + L(g,[a,b]) \le L(f+g,[a,b])$$

and

$$U(f+g,[a,b]) \le U(f,[a,b]) + U(g,[a,b]).$$

pf.: Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be bdd fncts. $\Longrightarrow (f+g) : [a,b] \longrightarrow \mathbb{R}$ is a bdd fnct.

From Real Analysis, for any set $A \subseteq \mathbb{R}$,

$\sup\limits_{x\in A}(f+g) \le \sup\limits_{x\in A}(f) + \sup\limits_{x\in A}(g)$ and $\inf\limits_{x\in A}(f+g) \ge \inf\limits_{x\in A}(f) + \inf\limits_{x\in A}(g)$. $(\star)$

By def. 1.3, the upper and lower Riemann sums over any partition P of $[a,b]$ are defined as follows:

$U(h,P,[a,b]) = \sum\limits_{j=1}^{n}(x_j - x_{j-1}) \sup\limits_{[x_{j-1},x_j]} h$ and $L(h,P,[a,b]) = \sum\limits_{j=1}^{n}(x_j - x_{j-1}) \inf\limits_{[x_{j-1},x_j]} h$, where $h: [a,b] \to \mathbb{R}$ is a bdd fnct. $(*)$

Then, combining $(\star)$ and $(*)$ and substituting $h = f, g, (f+g)$ provides

$U(f+g,P,[a,b]) \le U(f,P,[a,b]) + U(g,P,[a,b])$ and

$L(f+g,P,[a,b]) \ge L(f,P,[a,b]) + L(g,P,[a,b])$, for any partition P. (•)

By def. 1.7, the upper and lower Riemann integrals are defined as follows:

$U(h,[a,b]) = \inf\limits_{\text{all } P} U(h,P,[a,b])$ and $L(h,[a,b]) = \sup\limits_{\text{all } P} L(h,P,[a,b])$, where $h: [a,b] \to \mathbb{R}$ is a bdd fnct.

$\Longrightarrow U(h,[a,b]) \le U(h,P,[a,b])$ and $L(h,[a,b]) \ge L(h,P,[a,b])$, for any partition P, and these hold for $h = f, g, (f+g)$. (•)

Let $\varepsilon > 0$ be arbitrary. It follows from the defs of infimum and supremum

that, for $\varepsilon > 0$, there exist partitions P_1, P_2, P_3, and P_4 of $[a,b]$ s.t.

$U(f,P_1,[a,b]) \le U(f,[a,b]) + \frac{\varepsilon}{2}$ and $U(g,P_2,[a,b]) \le U(g,[a,b]) + \frac{\varepsilon}{2}$.

Likewise, $L(f,P_3,[a,b]) \ge L(f,[a,b]) - \frac{\varepsilon}{2}$ and $L(g,P_4,[a,b]) \ge L(g,[a,b]) - \frac{\varepsilon}{2}$. (•)

Let $P = P_1 \cup P_2 \cup P_3 \cup P_4$. Then, by thm. 1.5, since $P_1, P_2, P_3, P_4 \subseteq P$,

$U(f,P,[a,b]) \le U(f,P_1,[a,b])$ and $U(g,P,[a,b]) \le U(g,P_2,[a,b])$.

Similarly, $L(f,P,[a,b]) \ge L(f,P_3,[a,b])$ and $L(g,P,[a,b]) \ge L(g,P_4,[a,b])$. (•)

Putting (•), (•), (•), and (•) all together yields

$U(f+g,[a,b]) \le U(f+g,P,[a,b]) \le U(f,P,[a,b]) + U(g,P,[a,b]) \le U(f,P_1,[a,b]) + U(g,P_2,[a,b]) \le U(f,[a,b]) + U(g,[a,b]) + \varepsilon$ and

$L(f+g,[a,b]) \ge L(f+g,P,[a,b]) \ge L(f,P,[a,b]) + L(g,P,[a,b]) \ge L(f,P_3,[a,b]) + L(g,P_4,[a,b]) \ge L(f,[a,b]) + L(g,[a,b]) - \varepsilon$.

Simplifying gives us $U(f+g,[a,b]) \le U(f,[a,b]) + U(g,[a,b]) + \varepsilon$

and $L(f+g,[a,b]) \ge L(f,[a,b]) + L(g,[a,b]) - \varepsilon$, $\forall \varepsilon > 0$.

$\therefore$ Since this holds for all $\varepsilon > 0$, we conclude that

$U(f+g,[a,b]) \le U(f,[a,b]) + U(g,[a,b])$ and $L(f+g,[a,b]) \ge L(f,[a,b]) + L(g,[a,b])$. //