# HW9 MTH 650 Fall 2025

Sunday, November 16, 2025 3:42 PM

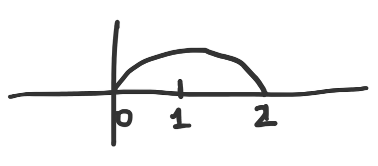
## §3B #6

**6** Let $\lambda$ denote Lebesgue measure on $\mathbf{R}$. Give an example of a continuous function $f\colon [0,\infty) \to \mathbf{R}$ such that $\lim_{t\to\infty} \int_{[0,t]} f\, d\lambda$ exists (in $\mathbf{R}$) but $\int_{[0,\infty)} f\, d\lambda$ is not defined.

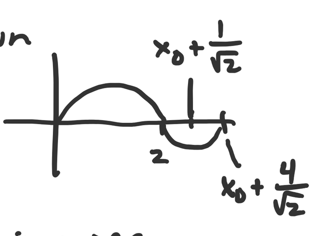
$\left|\int f\right| \le \int |f|$

Soln: Imagine semicircles w/ radius $\frac{1}{\sqrt{k}}$:


odd k → positive { Place the one for $k=1$ with center at 1:




even k → negative { Call $x_0 = 2$. For $k=2$, place semicircle w/ radius $\frac{1}{\sqrt{2}}$ upside-down at point $x_0 + \frac{1}{\sqrt{2}}$. Call right vertex $x_1 = x_0 + 2\left(\frac{2}{\sqrt{2}}\right)$ :




Continue placing circle w/ radius $\frac{1}{\sqrt{j+1}}$ upside-down from the previous one at point $x_j + \frac{1}{\sqrt{j+1}}$. Call right vertex $x_{j+1} = x_j + 2\left(\frac{1}{\sqrt{j+1}}\right)$.

Each semi-circle w/ radius $\frac{1}{\sqrt{j}}$ has area $\frac{\pi}{2j}$.

Call this function $f$.

Then, for any $N = 1, 2, \dots$

$$\int_{[0,x_N]} f\, d\lambda = \sum_{k=1}^{N} \frac{(-1)^k \pi}{2k} = \frac{\pi}{2}\sum_{k=1}^{N} \frac{(-1)^k}{k}$$

and so

$$\lim_{N\to\infty} \int_{[0,x_N]} f\, d\lambda = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$ , which is a convergent series by the alternating series test.

However,

$$\int_{[0,\infty)} f\, d\lambda = \int_{[0,\infty)} f^+ d\lambda - \int_{[0,\infty)} f^- d\lambda$$

and both

odd k → positive: $$\int f^+ d\lambda = \sum_{k=1}^{\infty} \frac{\pi}{2(2k+1)} = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{2k+1}$$ diverges by limit comparison test with harmonic series

and

even k → negative: $$\int f^- d\lambda = \sum_{k=1}^{\infty}\frac{\pi}{2(2k)} = \frac{\pi}{4}\sum_{k=1}^{\infty}\frac{1}{k}$$ diverges since it is the harmonic series

But def 3.18 does not permit us to define $\int_{[0,\infty)} f\, d\lambda$ unless one of $\int f^+ d\lambda$ or $\int f^- d\lambda$ is finite.

Thus we have shown

$$\lim_{t\to\infty} \int_{[0,t]} f\, d\lambda \equiv \lim_{N\to\infty} \int_{[0,N]} f\, d\lambda = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

while

$\int_{[0,\infty)} f\, d\lambda$ does not exist.

## #11

**11** Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mu)$. Prove that the set $\{x \in X : f(x) \neq 0\}$ is the countable union of sets with finite $\mu$-measure.

Proof: Let $E = \{x \in X : f(x) \neq 0\}$. Since $f \in \mathcal{L}^1(\mu)$, we know that

$$\int |f|\, d\mu < \infty.$$

Let $E_n = \{x \in X : |f(x)| \ge \frac{1}{n}\}$. Clearly, $E = \bigcup_{n=1}^{\infty} E_n$.

It remains to show that $\mu(E_n)$ is finite.

Since $f \ge \frac{1}{n}$ on $E_n$,

$$\infty > \int |f|\, d\mu > \int_{E_n} |f|\, d\mu \ge \int_{E_n} \frac{1}{n}\, d\mu = \frac{1}{n}\int_{E_n} 1\, d\mu$$
$$= \frac{1}{n}\int \chi_{E_n}\, d\mu$$
$$= \frac{\mu(E_n)}{n}$$

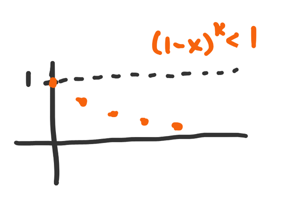
Thus we conclude that $\mu(E_n) < \infty$, completing the proof. ∎

## #12

**12** Suppose

$$f_k(x) = \frac{(1-x)^k \cos\frac{k}{x}}{\sqrt{x}}.$$

Prove that $\lim_{k\to\infty} \int_0^1 f_k = 0$.




Proof: Since for $x \in (0,1)$,

$$\frac{(1-x)^k \cos\left(\frac{k}{x}\right)}{\sqrt{x}} \le \frac{(1-x)^k}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$$

and we know that

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{b\to 0^+} \int_b^1 x^{-1/2}\, dx = \lim_{b\to 0^+} \left.\frac{x^{1/2}}{1/2}\right|_b^1$$ (See next problem!)

$$= 2\lim_{b\to 0^+} 1 - \sqrt{b} = 2$$

So the functions $f_k(x) = \frac{(1-x)^k \cos\left(\frac{k}{x}\right)}{\sqrt{x}}$ are measurable and converge pointwise to the zero function $f(x) = 0$.

Also if $g(x) := \frac{1}{\sqrt{x}}$ we know $\int_{(0,1)} g\, d\lambda < \infty$ and $|f_k| \le |g|$, so by dominated convergence thm, we conclude that

$$\lim_{k\to\infty} \int f_k\, d\lambda = \int \lim_{k\to\infty} f_k\, d\lambda = \int 0\, d\lambda = 0$$

## #14

**14** Let $\lambda$ denote Lebesgue measure on $\mathbf{R}$.

(a) Let $f(x) = 1/\sqrt{x}$. Prove that $\int_{[0,1]} f\, d\lambda = 2$.

(b) Let $f(x) = 1/(1+x^2)$. Prove that $\int_{\mathbf{R}} f\, d\lambda = \pi$.

(c) Let $f(x) = (\sin x)/x$. Show that the integral $\int_{(0,\infty)} f\, d\lambda$ is not defined but $\lim_{t\to\infty} \int_{(0,t)} f\, d\lambda$ exists in $\mathbf{R}$.

Soln (a): Let $E_n = [\frac{1}{n}, 1]$ for $n = 1, 2, \dots$ Let $f_n(x) = \frac{\chi_{[\frac{1}{n},1]}(x)}{\sqrt{x}}$

Clearly,

$$\lim_{n\to\infty} f_n(x) = \frac{1}{\sqrt{x}}$$ and $f_1 \le f_2 \le \dots$ is an increasing seq. of msbl. functions

So,

$$\int_{[0,1]} f\, d\lambda = \int_{[0,1]} \lim_{n\to\infty} f_n\, d\lambda \underset{\substack{\text{monotone}\\ \text{conv thm}\\ \text{Thm 3.11}}}{=} \lim_{n\to\infty} \int_{[0,1]} f_n\, d\lambda \underset{\substack{\text{def of } f_n \\ \text{and fact that Lebesgue} \\ \text{and Riemann } \int \\ \text{agree here} \\ \text{(Thm 3.34)}}}{=} \lim_{n\to\infty} \int_{[0,1]} \frac{\chi_{[\frac{1}{n},1]}(x)}{\sqrt{x}}\, dx = \lim_{n\to\infty} \int_{\frac{1}{n}}^{1} x^{-1/2}\, dx$$

$$= \lim_{n\to\infty} \left[ \left. 2\sqrt{x} \right|_{x=\frac{1}{n}}^{x=1} \right]$$
$$= \lim_{n\to\infty} \left[ 2 - 2\sqrt{\tfrac{1}{n}} \right]$$
$$= 2$$

Soln (b): Let $E_n = [-n, n]$ for $n = 1, 2, \dots$ and let

$$f_n(x) = \frac{\chi_{E_n}(x)}{x^2+1}$$

Clearly $f_n$ converges pointwise to $f(x) = \frac{1}{1+x^2}$

and $|f_n| \le f$.

Thus

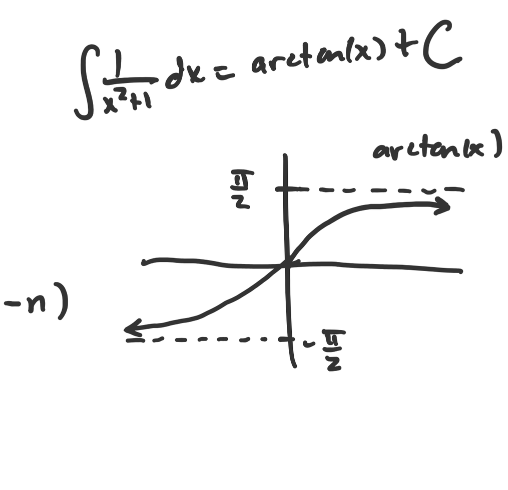
$$\int_{\mathbb{R}} f\, d\lambda = \int \lim_{n\to\infty} f_n\, d\lambda \underset{\text{monotone conv thm, def of } f_n\text{, Thm 3.34}}{=} \lim_{n\to\infty} \int \frac{\chi_{E_n}(x)}{x^2+1}\, dx$$

$$= \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{x^2+1}\, dx$$

$$= \lim_{n\to\infty} \arctan(n) - \arctan(-n)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$$\int \frac{1}{x^2+1}\, dx = \arctan(x) + C$$




## §4A #1

**1** Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h\colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Prove that

$$\mu(\{x \in X : |h(x)| \ge c\}) \le \frac{1}{c^p} \int |h|^p\, d\mu$$

for all positive numbers $c$ and $p$.

taking power is order-preserving: $x > y \Rightarrow x^p > y^p$

Proof: Let $E = \{x \in X : |h(x)| \ge c\}$ and let $p > 0$. So $c^p \le |h(x)|^p$.

$$\mu(E) = \frac{c^p}{c^p} \int \chi_E\, d\mu$$

$$= \frac{1}{c^p} \int_E c^p\, d\mu$$

$c^p \le |h(x)|^p$ for $x \in E$

$$\le \frac{1}{c^p} \int |h(x)|^p\, d\mu,$$

completing the proof. ∎

## #2

**2** Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) = 1$ and $h \in \mathcal{L}^1(\mu)$. Prove that

$$\mu\left(\left\{x \in X : \left|h(x) - \int h\, d\mu\right| \ge c\right\}\right) \le \frac{1}{c^2}\left(\int h^2\, d\mu - \left(\int h\, d\mu\right)^2\right)$$

for all $c > 0$.

[*The result above is called Chebyshev's inequality; it plays an important role in probability theory. Pafnuty Chebyshev (1821–1894) was Markov's thesis advisor.*]

Proof: Let $\alpha = \int h\, d\mu$.

Let $E = \{x \in X : |h(x) - \alpha| \ge c\}$.

Squaring, we get

$$E = \{x \in X : (h(x) - \alpha)^2 \ge c^2\}$$

Since $(h(x) - \alpha)^2$ is non-negative function, Markov's inequality (Thm 4.1) shows that

$$\mu(E) \le \frac{1}{c^2} \|(h-\alpha)^2\|_1 = \frac{1}{c^2} \int (h-\alpha)^2\, d\mu$$

$$= \frac{1}{c^2}\Big[\int h^2\, d\mu \underbrace{- 2\alpha \underset{=\alpha \text{ by def of } \alpha}{\int h\, d\mu} + \alpha^2 \overset{=\mu(X)=1, \text{ by hypothesis}}{\int 1\, d\mu}}_{=-2\alpha^2+\alpha^2=-\alpha^2}\Big]$$

$$= \frac{1}{c^2}\left[\int h^2\, d\mu - \alpha^2\right]$$

$$\overset{\text{def of } \alpha}{=} \frac{1}{c^2}\left[\int h^2\, d\mu - \left(\int h\, d\mu\right)^2\right].$$ ∎