HW8 MTH 650 Fall 2025

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§3A #9

Suppose μ is a measure on a measurable space $(X, \mathcal{S})$ and $f\colon X \to [0,\infty]$ is an $\mathcal{S}$-measurable function. Define $\nu\colon \mathcal{S} \to [0,\infty]$ by

$$\nu(A) = \int \chi_A f \, d\mu$$

for $A \in \mathcal{S}$. Prove that ν is a measure on $(X, \mathcal{S})$.

Proof: NTS (i) $\nu: \mathcal{S} \to [0,\infty]$ (i.e. ν is non-negative)

(ii) $\nu(\emptyset) = 0$, and

(iii) for any disjoint $A_1, \dots, A_n \in \mathcal{S}$,

$$\nu(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} \mu(A_k)$$

Thm 3.8

(i) First, since $0 \le f$, we know for any $A \in \mathcal{S}$, $\int f \chi_A \, d\mu \geq 0$. Thus $\nu: \mathcal{S} \to [0,\infty]$.

(ii) Since μ is a measure, $\mu(\emptyset) = 0$. Now compute

$$\nu(\emptyset) \overset{\text{def}}{=} \int \chi_\emptyset f \, d\mu = \int 0 \, d\mu = 0$$

$\chi_\emptyset$ is the zero function!

(iii) Now suppose $A_1, \dots, A_n \in \mathcal{S}$ are disjoint.

Notice

(*) $$\chi_{A_1 \cup \dots \cup A_n}(x) = \begin{cases} 1, & x \in A_1 \cup \dots \cup A_n \\ 0, & x \notin X \setminus A_1 \cup \dots \cup A_n \end{cases} = \begin{cases} 1, & x \in A_1 \text{ or } x \in A_2 \text{ or} \dots \text{or } x \in A_n \\ 0, & x \in X \setminus (A_1 \cup \dots \cup A_n) \end{cases}$$

$$= \chi_{A_1}(x) + \dots + \chi_{A_n}(x)$$

Therefore,

$$\nu(A_1 \cup \dots \cup A_n) \overset{\text{def}}{=} \int \chi_{A_1 \cup \dots \cup A_n} f$$

$$\overset{(*)}{=} \int [\chi_{A_1} + \chi_{A_2} + \dots + \chi_{A_n}] f$$

$$\overset{\text{Thm 3.21}}{=} \int \chi_{A_1} f + \int \chi_{A_2} f + \dots + \int \chi_{A_n} f$$

$$\overset{\text{def}}{=} \nu(A_1) + \nu(A_2) + \dots + \nu(A_n),$$

completing the proof that ν is a measure. ∎

recall an inf sum $\sum_{k=1}^{\infty} \beta_k$ is defined by its partial sum as a limit:

$$\sum_{k=1}^{\infty} \beta_k \overset{\text{def}}{=} \lim_{j\to\infty} \sum_{k=1}^{j} \beta_k$$

#10

10 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f_1, f_2, \dots$ is a sequence of nonnegative $\mathcal{S}$-measurable functions. Define $f\colon X \to [0,\infty]$ by $f(x) = \sum_{k=1}^{\infty} f_k(x)$. Prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Proof: By def,

(*) $$f(x) = \sum_{k=1}^{\infty} f_k(x) = \lim_{j\to\infty} \sum_{k=1}^{j} f_k(x)$$

Notice that for any finite M,

(**) $$\sum_{k=1}^{M} \int f_k \, d\mu = \int \Big(\sum_{k=1}^{M} f_k\Big) d\mu$$

So by def,

(***) $$\sum_{k=1}^{\infty} \int f_k \, d\mu = \lim_{j\to\infty} \sum_{k=1}^{j} \int f_k \, d\mu \overset{(**)}{=} \lim_{j\to\infty} \int \Big(\sum_{k=1}^{j} f_k\Big) d\mu$$

Therefore, we need to show:

$$\int \lim_{j\to\infty} \sum_{k=1}^{j} f_k(x) \, d\mu \overset{(*)}{=} \int f \overset{\boxed{\text{goal}}}{=} \sum_{k=1}^{\infty} \int f_k \, d\mu \overset{(***)}{=} \lim_{j\to\infty} \int \Big(\sum_{k=1}^{j} f_k\Big) d\mu$$

Let $g_j = \sum_{k=1}^{j} f_k$. So, we need to show for the function $g := \lim_{j\to\infty} g_j$, that

(†) $$\int g \, d\mu = \lim_{j\to\infty} \int g_j \, d\mu.$$

Since the f_k are non-negative, we see that $g_1 \le g_2 \le g_3 \le \dots$. Therefore, we may apply the monotone convergence thm (Thm 3.11) to conclude (†), completing the proof. ∎

#13

13 Give an example to show that the Monotone Convergence Theorem (3.11) can fail if the hypothesis that $f_1, f_2, \dots$ are nonnegative functions is dropped.

> **3.11 *Monotone Convergence Theorem*** (MCT)
>
> Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $0 \le f_1 \le f_2 \le \cdots$ is an increasing sequence of $\mathcal{S}$-measurable functions. Define $f\colon X \to [0,\infty]$ by
>
> $$f(x) = \lim_{k\to\infty} f_k(x).$$
>
> Then
>
> $$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Suppose that we take $X = [0,\infty]$ and μ the Lebesgue measure.

Let $f_n = -\frac{x}{n}$. Since

$$-1 \le -\tfrac{1}{2} \le -\tfrac{1}{3} \le \cdots,$$

we get

$$f_1 \le f_2 \le f_3 \le \cdots,$$

satisfying a requirement for MCT.

Let $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} -\frac{x}{n} = 0$, the zero function.

Then

$$\int f \, d\mu = \int 0 \, d\mu = 0$$

But each

$$\int f_n \, d\mu = \int -\frac{x}{n} \, d\mu \Big(\equiv \int_0^{\infty} \frac{-x}{n} \, dx \Big) = -\infty$$

So we have

$$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} -\infty = -\infty \neq 0 = \int 0 \, d\mu = \int \lim_{n\to\infty} f_n = \int f,$$

contrary to conclusion of MCT.

§3B #3

3 Suppose λ is Lebesgue measure on $\mathbf{R}$ and $f\colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that $\int |f| \, d\lambda < \infty$. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \int_{(-\infty, x)} f \, d\lambda.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

means $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. if x, y obey $|x-y| < \delta$, then $|g(x) - g(y)| < \epsilon$

Proof: NTS

$\forall \epsilon > 0 \; \exists \delta > 0$ if $|x-y| < \delta$, then $|g(x) - g(y)| < \epsilon$

Wlog assume $x > y$, so $(-\infty, x) = (-\infty, y) \cup \{y\} \cup (y, x)$

$$\int_{-\infty}^{x} - \int_{-\infty}^{y} = \int_{-\infty}^{y} + \int_{y}^{x} - \int_{-\infty}^{y} = \int_{y}^{x}$$

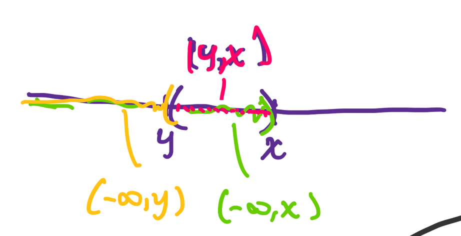
(★)
$$|g(x) - g(y)| = \Big| \int_{(-\infty,x)} f \, d\lambda - \int_{(-\infty,y)} f \, d\lambda \Big|$$

$$= \Big| \int_{(-\infty,y)\cup\{y\}\cup(y,x)} f \, d\lambda - \int_{(-\infty,y)} f \, d\lambda \Big|$$

$$= \Big| \cancel{\int_{(-\infty,y)} f \, d\lambda} + \underbrace{\int_{\{y\}} f \, d\lambda}_{=0} + \int_{(y,x)} f \, d\lambda - \cancel{\int_{(-\infty,y)} f \, d\lambda} \Big|$$

since $\lambda(\{y\}) = 0$ since λ is Lebesgue measure

$$= \Big| \int_{(y,x)} f \, d\lambda \Big| \le \int_{(y,x)} |f| \, d\lambda$$



Thm 3.28

$\int g \, d\mu < \infty \to \forall \epsilon > 0 \; \exists \delta > 0 \; \forall B \in \mathcal{S}$ w/ $\mu(B) < \delta$, $\int_B g \, d\mu < \epsilon$

Let $\epsilon > 0$. By Thm 3.28 we know $\exists \delta > 0$ so that $\forall B \in \mathcal{S}$ with $\mu(B) < \delta$,

$$\int_B |f| \, d\lambda < \epsilon$$

Thus, if $x - y < \delta$ (with $x > y$), then (x,y) is an open interval (hence a Borel set) which obeys $\mu((x,y)) < \delta$. Thus

$$\epsilon > \Big| \int_{(x,y)} |f| \, d\lambda \Big| \overset{(★)}{=} |g(x) - g(y)|,$$

Completing the proof that g is uniformly continuous. ∎

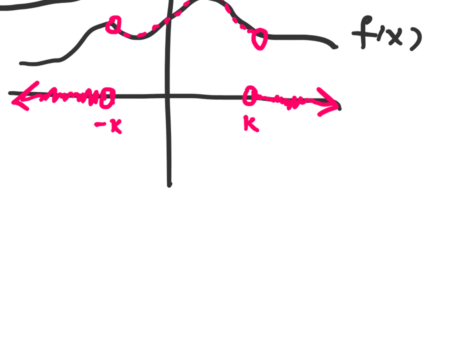
#5

5 Let λ denote Lebesgue measure on $\mathbf{R}$. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function such that $\int |f| \, d\lambda < \infty$. Prove that

$$\lim_{k\to\infty} \int_{[-k,k]} f \, d\lambda = \int f \, d\lambda.$$

note: f_k is msbl b/c f is msbl and the intervals $[-k,k]$ are Borel sets (hence by a Thm, $\chi_{[-k,k]}$ is Borel msbl)

Proof: Let $f_k: \mathbb{R} \to \mathbb{R}$ be defined as $f_k(x) = f(x)\chi_{[-k,k]}(x)$.



Clearly, $f(x) = \lim_{k\to\infty} f_k(x)$, so f_k converges pointwise to f.

Also,

(*) $$\int f_k \, d\lambda = \int f \chi_{[-k,k]} \, d\lambda = \int_{[-k,k]} f \, d\lambda$$

Since $\int |f| \, d\lambda < \infty$, we know that by def of $\chi_{[-k,k]}$ that

$$|f_k| = |f \chi_{[-k,k]}| \le |f|,$$

which holds for every $k = 1, 2, 3, \dots$.

Therefore, by dominated conv thm (Thm 3.31), since we know

1. each f_k is measurable,
2. the f_k converge pointwise to f, and
3. the function $|f|$ is msbl and $\int |f| < \infty$ and $|f_k| \le |f|$ for all k and all x,

that we can conclude that

$$\lim_{k\to\infty} \int_{[-k,k]} f \, d\lambda \overset{(*)}{=} \lim_{k\to\infty} \int f_k \, d\lambda = \int f \, d\lambda,$$

Completing the proof. ∎