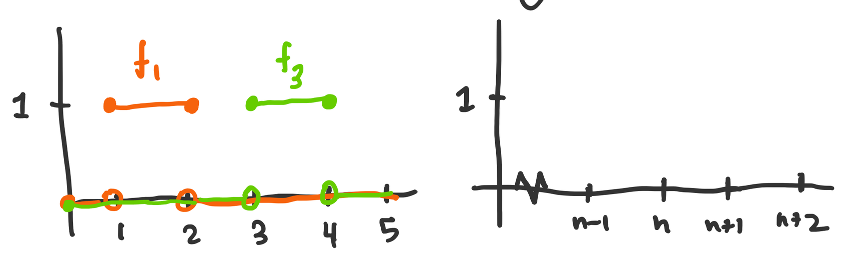
HW7 MTH 650 Fall 2025

Sunday, November 16, 2025 9:14 AM

§2E #5 Consider μ to be Lebesgue measure on $\mathbb{R}^+ \equiv [0,\infty)$.

Let $f_n : \mathbb{R}^+ \to \{0,1\}$ be $f_n(x) = \chi_{[n,n+1]}(x) = \begin{cases} 1, & x \in [n,n+1] \\ 0, & x \in \mathbb{R}^+ \setminus [n,n+1] \end{cases}$



Clearly $\{f_n\}$ converges pointwise to the function $f : \mathbb{R}^+ \to \{0\}$, $f(x) = 0$.

Suppose that $\exists\, E \subseteq \mathbb{R}^+$ satisfying Egorov thm so that $\{f_n\}$ converges uniformly on E, i.e.

Egorov also requires $\mu(\mathbb{R}^+ \setminus E) < \epsilon$, which we state below

$$\forall \epsilon > 0\ \exists N\ \forall n > N\ \forall x \in E,\ |f_n(x) - f(0)| \equiv |f_n(x)| < \epsilon$$

Choose $\epsilon = 1$. By our suppositions, $\mu(\mathbb{R}^+ \setminus E) < 1$ and

$$\exists N\ \forall n > N\ \forall x \in E\quad |f_n(x)| < 1.$$

The only x that can satisfy $|f_n(x)| < 1$ for all sufficiently large n are those x lying in $(0,N]$. Therefore $E \subseteq (0,N]$.

But this means $\boxed{\mu(\mathbb{R} \setminus E) = \infty}$, which contradicts our assumption that $\mu(\mathbb{R}^+ \setminus E) < 1$.

(Dini's theorem)

#7 Let $F \subseteq \mathbb{R}$ be closed and bounded (F is compact) and $g_n : F \to \mathbb{R}$ be so that they are continuous $\forall x \in F$

$g_1(x) \le g_2(x) \le g_3(x) \le \dots$

such that $\forall x \in F$, $\sup\{g_1(x), g_2(x), \dots\} < \infty$.

Define $g : F \to \mathbb{R}$ by $g(x) = \lim_{k\to\infty} g_k(x)$.

Goal: g is continuous iff $g_1, g_2, \dots$ conv uniformly on F to g

Proof: $(\rightarrow)$ Assume g ctn.

NTS: $\forall \epsilon > 0\ \exists N\ \forall n > N\ \forall x \in F\ |g_n(x) - g(x)| < \epsilon$

Since the g_n converge pointwise to g,

$$\forall x \in F\ \forall \epsilon > 0\ \exists N_x\ \forall n > N_x\ |g_n(x) - g(x)| < \epsilon$$

Since the g_n are continuous,

$$\forall x \in F\ \forall \epsilon > 0\ \exists \delta_x > 0 \text{ if } s \in (x-\delta_x, x+\delta_x), \text{ then } |g_n(x) - g_n(s)| < \epsilon$$

Let $I_{\delta_x} = (x-\delta_x, x+\delta_x)$. Then $\bigcup_{x\in F} I_{\delta_x}$ is an open cover of F.

Since F is compact, there is a finite subcover $\theta := \bigcup_{n=1}^{M} I_{\delta_{x_n}}$ of F.

Let $\epsilon > 0$ and $N = \max\{N_{x_1}, N_{x_2}, \dots, N_{x_M}\}$, so for any $n > N$ we know

$$\forall x \in F\quad |g_n(x) - g(x)| < \epsilon$$

$(\leftarrow)$ Follows from Theorem 2.84. ∎

#9 Spz $F_1, \dots, F_n$ disjoint closed subsets of $\mathbb{R}$.

Prove: if $g : F_1 \cup \dots \cup F_n \to \mathbb{R}$ is a function such that $g|_{F_k}$ is ctn for each $k = 1, \dots, n$, then g is continuous

Proof: Let $g_k \equiv g|_{F_k}$

Need to show

$$\forall x \in F_1 \cup \dots \cup F_n\ \forall \epsilon > 0\ \exists \delta_x > 0 \text{ if } s \in (x-\delta_x, x+\delta_x) \cap (F_1 \cup \dots \cup F_n), \text{ then } |g(x) - g(s)| < \epsilon$$

We know for each $k = 1, 2, \dots, n$ that

$$\forall x \in F_k\ \exists \delta_{k_x} > 0 \text{ if } s \in (x-\delta_{k_x}, x+\delta_{k_x}) \cap F_k, \text{ then } |g_k(x) - g_k(s)| < \epsilon$$

Let $x \in F_1 \cup \dots \cup F_n$, so choose $\delta = \min\{\delta_{1_x}, \delta_{2_x}, \dots, \delta_{k_x}\}$. Then, if $s \in (x-\delta, x+\delta) \cap (F_1 \cup \dots \cup F_n)$, we have $|g(x) - g(s)| < \epsilon$. ∎

#10 Spz $F \subseteq \mathbb{R}$ has property that every continuous function $g : F \to \mathbb{R}$ can be extended to a continuous function $h : \mathbb{R} \to \mathbb{R}$ (i.e. h is ctn and for all $x \in F$, $h(x) = g(x)$).

Prove F is closed.

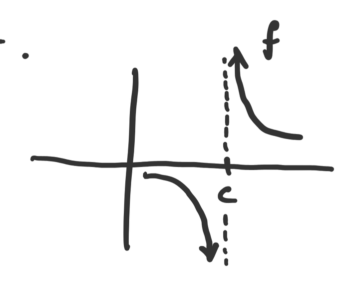
Proof: Suppose F is not closed, so it does not contain all of its limit points.

Let $c \in \mathbb{R}$ be a limit point of F such that $c \notin F$.

So there is a sequence $x_n \in F$ so that $x_n \to c$.

Consider the function $f : F \to \mathbb{R}$ given by $f(x) = \frac{1}{x-c}$.

Clearly f is continuous on F (since $c \notin F$).



We cannot extend f to a continuous function h because $\lim_{x\to c} f(x)$ does not exist in $\mathbb{R}$!

§3A #1 Spz $(X, \mathcal{S}, \mu)$ is measure space and $f : X \to [0,\infty]$ is $\mathcal{S}$-msbl function such that $\int f\, d\mu < \infty$. Explain why $\inf_E f = 0$ for each $E \in \mathcal{S}$ with $\mu(E) = \infty$.

Soln: Suppose there were an $\mathcal{S} \in E$ with $\mu(E) = \infty$ and $\inf_E f = W > 0$.

In this case

$$\int f\, d\mu \ge \int f \chi_E\, d\mu \ge \int W \chi_E\, d\mu$$

$f \ge 0$ and $E \subseteq X$

prop of $\int$

$$= W \int \chi_E\, d\mu$$

$$= W \mu(E) = \infty,$$

Fact about integrating χ_E

which is a contradiction

#2 **2** Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $c \in X$. Define the Dirac measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

Prove that if $f : X \to [0,\infty]$ is $\mathcal{S}$-measurable, then $\int f\, d\delta_c = f(c)$.
[*Careful:* $\{c\}$ *may not be in* $\mathcal{S}$.]

Proof: Suppose $f : X \to [0,\infty]$ is $\mathcal{S}$-msbl.

Then for all $B \in \mathcal{S}$, $f^{-1}(B) \in \mathcal{S}$.

By Thm 3.9,

$$(\star)\qquad \int f\, d\delta_c = \sup\left\{ \sum c_j \mu(A_j) : \begin{array}{l} \{A_1, \dots, A_j\} \text{ is an } \mathcal{S}\text{-partition}, \\ c_1, \dots, c_j \in [0,\infty),\ f(x) \ge \sum c_j \chi_{A_j} \end{array} \right\}$$

So for any $\mathcal{S}$-partition $A_1, \dots, A_j$ of X, we know it is disjoint and

$$X = \bigcup_{k=1}^{n} A_k \text{ (disjoint union), so } \exists \text{ some } A_\ell \text{ s.t. } c \in A_\ell.$$

Consider a simple function $g(x) = \sum c_j \chi_{A_j}$. Then,

$$\int g(x)\, d\delta_c = \sum c_j \delta_c(A_j) \underset{\text{def of } \delta_c}{=} c_\ell, \quad c_\ell \le f(c).$$

Suppose we take $c_\ell = f(c)$. Notice this is largest c_ℓ can be!

So, $\int g\, d\delta_c = f(c)$.

Thus we get

$$\int f\, d\delta_c \underset{(\star)}{\ge} \int \sum c_j \chi_{A_j} = \sum c_j \delta_c(A_j) = c_\ell = f(c)$$

Also,

$$f(c) = f(c)\underbrace{\chi_{A_\ell}(x)}_{=0 \text{ except at } x \in A_\ell} \le \overbrace{\sum f(c) \chi_{A_j}}^{\text{more non-neg terms}} \underset{\text{choose } c_j \ge f(c)}{\le} \sum c_j \chi_{A_j}$$

Now taking sup over all $\mathcal{S}$-partitions P and constants $c_1, \dots, c_n$ while keeping $\sum c_j \chi_{A_j}(x) \le f(x)$ gives

$$f(c) \le \int f\, d\delta_c.$$

Thus we have

$$\int f\, d\delta_c \le f(c) \le \int f\, d\delta_c,$$

completing the proof. ∎

#3 **3** Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f : X \to [0,\infty]$ is an $\mathcal{S}$-measurable function. Prove that

$$\int f\, d\mu > 0 \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0.$$

Proof: Let $E = \{x \in X : f(x) > 0\}$.

$(\rightarrow)$ Assume (for contrapositive) that $\mu(E) = 0$.

Since

$$\int f\, d\mu \underset{\text{Thm 3.9}}{=} \sup\left\{ \int g\, d\mu,\ g \text{ simple w/ } g = \sum c_j \chi_{A_j};\ g \le f \right\}$$

Any such g can be nonzero only on subsets of E.

Thus

$$\int g\, d\mu = \sum c_j \mu(A_j) = 0$$

since any A_k where $c_j > 0$ is a subset of E and $A_k \subseteq E \Rightarrow \mu(A_k) \le \mu(E) = 0$, by assumption.

Thus

$$\int f\, d\mu = \sup\left\{\int g\right\} = \sup\{0\} = 0,$$

proving this direction.

$(\leftarrow)$ Suppose $\mu(E) > 0$. Consider sequence

$$E_n = \{x \in X : f(x) \ge \tfrac{1}{n}\},\ n = 1, 2, 3, \dots$$

Clearly,

$$E = \bigcup_{k=1}^{\infty} E_k$$

Since $\mu(E) > 0$ $\exists N$ s.t. $\mu(E_N) > 0$.

Now,

$$\int f \chi_{E_N}\, d\mu \ge \int \frac{1}{N} \chi_{E_N} = \frac{1}{N} \mu(E_N)$$

Thus,

$$\int f\, d\mu \ge \int f \chi_{E_N}\, d\mu = \frac{1}{N} \mu(E_N) > 0,$$

completing the proof. ∎