HW6 MTH 650 Fall 2025

Sunday, October 19, 2025 4:28 PM

§2D | #14 | Show $\frac{1}{4}$ and $\frac{9}{13}$ are in C

Pf:

Claim: $\frac{1}{4}$ in base 3 is $\frac{1}{4} = 0.020202\ldots_3$

Pf:

a number in $[0,1]$ in base 3 takes the form $\sum_{k=1}^{\infty} a_k 3^{-k}$ for $a_k \in \{0,1,2\}$

Notice that

$$\frac{1}{4} = \frac{2}{8} = \frac{2}{9-1} = \frac{2}{9(1-\frac{1}{9})} = \frac{2}{9}\frac{1}{1-\frac{1}{9}}$$

Why this? Because $9 = 3^2$ relates to base 3!

Now,

$$\frac{1}{1-\frac{1}{9}} = \sum_{k=0}^{\infty}\left(\frac{1}{9}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{3^2}\right)^k = 1 + 0.01010101\ldots_3$$

(geo series; digits correspond to $\frac{1}{3^2}, \frac{1}{3^4}, \frac{1}{3^6}, \ldots$)

Thus,

$$\frac{1}{4} = \frac{2}{9}\left(\frac{1}{1-\frac{1}{9}}\right) = \frac{2}{3^2}\left(1.010101\ldots_3\right)$$

$$= \frac{1}{3^2}\left(2.020202\ldots_3\right)$$

$$= 0.020202\ldots_3$$

(since $\frac{1}{3^2}$ moves decimal point left by 2 in base 3 just like $\frac{1}{10^2}$ moves it left by 2 in base 10)

$0.010101\ldots_3 \times 0.002_3$

Thus $\frac{1}{4}$ has decimal expansion using 0 and 2 in base 3. Thm 2.75 shows $\frac{1}{4} \in C$.

Similarly, (smallest power of 3 larger than 13)

n	3^n
1	3
2	9
3	27
4	

$$\frac{9}{13} = \frac{9}{27-14} = \frac{9}{27(1-\frac{14}{27})} = \frac{3^2}{3^3}\left(\frac{1}{1-\frac{14}{3^3}}\right) = \frac{1}{3}\left(\frac{1}{1-\frac{14}{3^3}}\right)$$

But

$14 = 9+5 = 9+3+2$

$$\frac{1}{1-\frac{14}{3^3}} = \sum_{k=0}^{\infty}\left(\frac{14}{27}\right)^k = \sum_{k=0}^{\infty}\left(\frac{9+3+2}{27}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{3}+\frac{1}{9}+\frac{2}{27}\right)^k$$

← difficult to analyze

Instead, think: we want $a_1, a_2, \ldots$ so that $a_i \in \{0,1,2\}$ and

$$\frac{9}{13} = 0.a_1a_2a_3\ldots_3 = \sum_{k=1}^{\infty}\frac{a_k}{3^k}$$

So mult by 3 to see

$$\frac{27}{13} = \sum_{k=1}^{\infty}\frac{a_k}{3^{k-1}} = a_1 + \sum_{k=2}^{\infty}\frac{a_k}{3^{k-1}}$$

(a_1: integer $\in \{0,1,2\}$)

But also

$$\frac{27}{13} = \frac{26+1}{13} = 2 + \frac{1}{13}$$

Thus $a_1 + \sum_{k=2}^{\infty}\frac{a_k}{3^{k-1}} = 2 + \frac{1}{13}$

So, $\boxed{a_1 = 2}$ and $\frac{1}{13} = \sum_{k=2}^{\infty}\frac{a_k}{3^{k-1}}$

Now, mult by 3 to get

$$0 + \frac{3}{13} = \frac{3}{13} = \sum_{k=2}^{\infty}\frac{a_k}{3^{k-2}} = a_2 + \sum_{k=3}^{\infty}\frac{a_k}{3^{k-2}}$$

$\Rightarrow \boxed{a_2 = 0}$

Now mult again by 3 to get

$$0 + \frac{9}{13} = \frac{9}{13} = a_3 + \sum_{k=4}^{\infty}\frac{a_k}{3^{k-3}} \Rightarrow \boxed{a_3 = 0}$$

Now mult by 3 again to get $\frac{27}{13}$

$$2 + \frac{1}{3} = \frac{27}{13} = a_4 + \sum_{k=5}^{\infty}\frac{a_k}{3^{k-4}}$$

$\Rightarrow \boxed{a_4 = 2}$

We have entered a Repetition!

So we have...

$a_1 = 2$
$a_2 = 0$
$a_3 = 0$
$a_4 = 2$

So $a_5 = a_6 = 0$, $a_7 = 2$, $a_8 = a_9 = 0, \ldots$, i.e.

$$\frac{9}{13} = 0.200200200200\ldots_3$$

Since the expansion has only 0's and 2's, $\frac{9}{13} \in C$.

#15 | Show $\frac{13}{17} \notin C$

Pf: Let

$$\frac{13}{17} = 0.a_1a_2a_3\ldots_3 = \sum_{k=1}^{\infty}\frac{a_k}{3^k}$$

Then mult by 3 to get

$$\frac{39}{17} = \frac{34+5}{17} = 2 + \frac{5}{17} \Rightarrow \boxed{a_1 = 2}$$

($34 = 17 \cdot 2$)

mult 3

$$\frac{15}{17} = 0 + \frac{15}{17} \Rightarrow \boxed{a_2 = 0}$$

mult 3

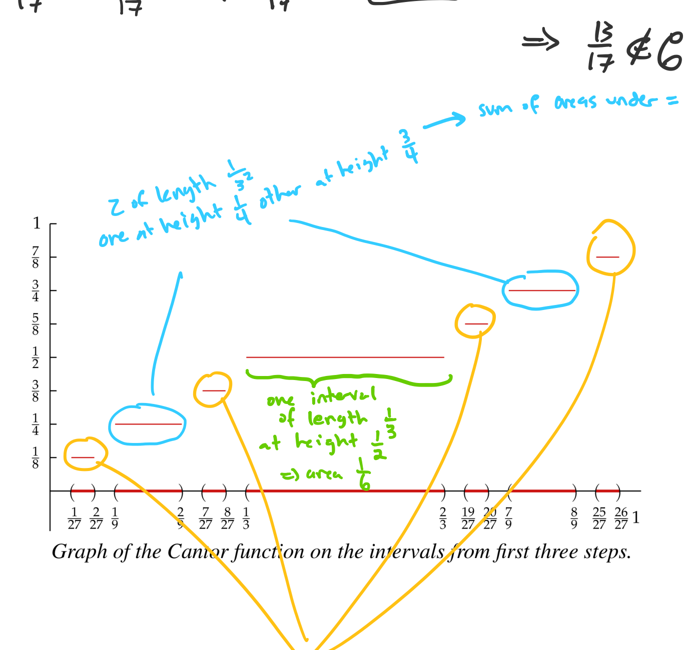
$$\frac{45}{17} = \frac{34+11}{17} = 2 + \frac{11}{17} \Rightarrow \boxed{a_3 = 2}$$

($34 = 17 \cdot 2$)

$$\frac{33}{17} = \frac{17+16}{17} = 1 + \frac{16}{17} \Rightarrow \boxed{a_4 = 1} \Longrightarrow 1 \text{ in base 3 decimal expansion of } \frac{13}{17}$$

$$\Rightarrow \frac{13}{17} \notin C$$

#19 |



Graph of the Cantor function on the intervals from first three steps.

4 of length $\frac{1}{3^3}$

$8 = 2^3 \rightarrow$ heights: $\frac{1}{2^3} + \frac{3}{2^3} + \frac{5}{2^3} + \frac{7}{2^3} = \frac{16}{8} = 2$

sum of areas of rectangles: $\frac{2}{3^3}$

Next step: 8 of length $\frac{1}{3^4}$

heights: $\frac{1}{2^4} + \frac{3}{2^4} + \frac{5}{2^4} + \frac{7}{2^4} + \frac{9}{2^4} + \frac{11}{2^4} + \frac{13}{2^4} + \frac{15}{2^4} = \frac{4}{2^4} = \frac{4(16)}{2^4} = 4$

sum of areas $= \frac{4}{3^4}$

In general there will be 2^n of length $\frac{1}{3^{n+1}}$ and sum of areas $= \frac{2^{n-1}}{3^{n+1}}$

Thus

$$\int_0^1 \Lambda = \sum_{k=0}^{\infty}\frac{2^{k-1}}{3^{k+1}} = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{6}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{6}\left(\frac{1}{1/3}\right) = \frac{3}{6} = \frac{1}{2}$$

(mult by $\frac{2}{2}$ and factor out one 3 from denom)

($k=0$ is the one representing the width $\frac{1}{3}$ height $\frac{1}{2}$ area!)

#20 | (a) $\Lambda\left(\frac{9}{13}\right) = \Lambda\left(0.200200\ldots_3\right)$ (by #14 … $\in C$)

$$\overset{\text{def 2.77}}{=} 0.100100100\ldots_2$$

(digits: $\frac{1}{2}, \frac{1}{2^4}, \ldots, \frac{1}{2^7}$)

$$= \sum_{k=0}^{\infty}\frac{1}{2^{3k+1}} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2^3}\right)^k$$

$$\overset{\text{geo series}}{=} \frac{1}{2}\left(\frac{1}{1-\frac{1}{2^3}}\right) = \frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{1}{2}\left(\frac{1}{7/8}\right) = \frac{8}{14} = \frac{4}{7}$$

(b) $\Lambda(0.93) = \Lambda\left(\frac{93}{100}\right) = \Lambda\left(0.2210022220 1\ldots_3\right)$ (truncate; $\notin C$ since 1 in expansion)

$$\overset{\text{def 2.77}}{=} 0.111_2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$$

#21 |

(a) $\Lambda^{-1}\left(\left\{\frac{1}{3}\right\}\right) \overset{\text{def of inverse}}{=} \left\{x \in [0,1] : \Lambda(x) = \frac{1}{3}\right\}$

$\frac{1}{3} = 0.010101\ldots_2$ ← not truncated!

⇓ change 1's to 2's + interpret in base 3

$$\Lambda^{-1}\left(\frac{1}{3}\right) = 0.020202\ldots_3 = \frac{1}{4}$$

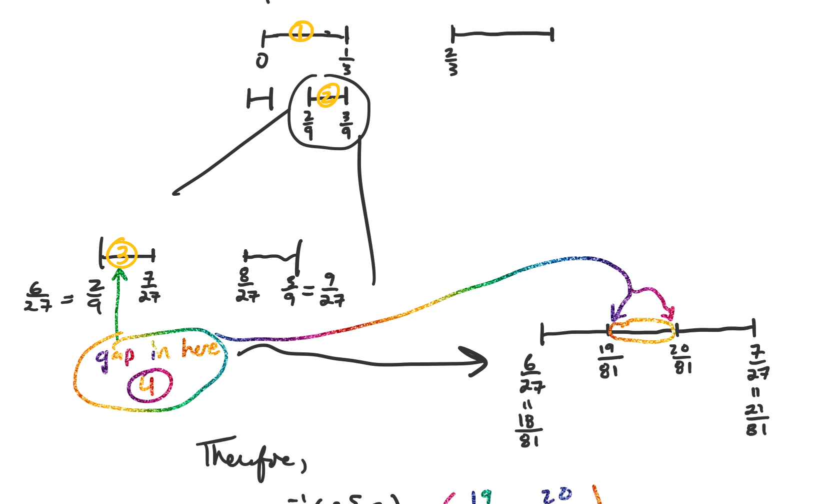
(b) $\Lambda^{-1}\left(\left\{\frac{5}{16}\right\}\right) = \Lambda^{-1}\left(0.0101_2\right)$ ← truncated! which interval is it on?

$\frac{5}{16} = \frac{4}{16} + \frac{1}{16} = \frac{1}{4} + \frac{1}{16} = 0.0101_2$

⇓ replace all but the last 1 with 2's + interpret in base 3

0.0201_3 (left on 1st step ①, right on 2nd step ②, left on 3rd step ③)

$= \frac{1}{9} + \frac{1}{81} = \frac{10}{81}$ → in gap of ④th step



Therefore,

$$\Lambda^{-1}\left(\left\{\frac{5}{16}\right\}\right) = \left(\frac{19}{81}, \frac{20}{81}\right)$$

§2E | #1 | Let $f_k : X \to \mathbb{R}$ converge pointwise and let X be finite set.

That means $\exists f : X \to \mathbb{R}$ s.t.

$$f(x) = \lim_{k\to\infty} f_k(x),$$

i.e. $\forall x \in X\ \forall \epsilon > 0\ \exists N\ \forall n > N\ |f_n(x) - f(x)| < \epsilon$.

WTS $f_k \to f$ uniformly, i.e.

$$\forall \epsilon > 0\ \exists N\ \forall n > N\ \forall x \in X\ |f_n(x) - f(x)| < \epsilon.$$

Let $\epsilon > 0$. Pointwise conv $\Rightarrow$ for any $x \in X$ $\exists N_x$ $\forall n > N_x$ $|f_n(x) - f(x)| < \epsilon$.

Let $N = \max\{N_x : x \in X\}$. Then for any $n > N$, $|f_n(x) - f(x)| < \epsilon$, which holds for all $x \in X$, completing the proof.

(necessarily finite!!!)

#2 | Find sequence $f_n : \mathbb{Z}^+ \to \mathbb{R}$ that converges ptwise but not uniformly.

Let $f_k(m) = \frac{k}{k+m}$

Then for each m, $\lim_{k\to\infty} f_k(m) = 1$ (define as $f(m)$), which is simple to prove: let $\epsilon > 0$ and choose $N > \left(\frac{1}{\epsilon} - 1\right)m$.

(notice our choice of N depends on m!!)

Then for $n > N$,

$$n > \left(\frac{1}{\epsilon} - 1\right)m \Rightarrow n > \frac{m}{\epsilon} - m$$

$$\Rightarrow n + m > \frac{m}{\epsilon} \rightarrow \frac{n+m}{m} > \frac{1}{\epsilon} \rightarrow \frac{m}{n+m} < \epsilon$$

$$|f_n(m) - 1| = \left|\frac{n}{n+m} - 1\right| = \left|\frac{m}{n+m}\right| < \epsilon.$$

$\left|\frac{n}{n+m} - 1\right| = \left|\frac{m}{n+m}\right| < \epsilon$

$\frac{m}{n+m} < \epsilon$

$\frac{n+m}{m} > \frac{1}{\epsilon}$

$n > \frac{m}{\epsilon} - m = \left(\frac{1}{\epsilon} - 1\right)m$

Suppose $f_n \to f$ uniformly, i.e. $\forall \epsilon > 0\ \exists N\ \forall n > N\ \forall m \in \mathbb{Z}^+\ |f_n(m) - f(m)| < \epsilon$

This means $\forall m \in \mathbb{Z}$

$$\left|\frac{n}{m+n} - 1\right| < \epsilon$$

$$\frac{m}{m+n} < \epsilon$$

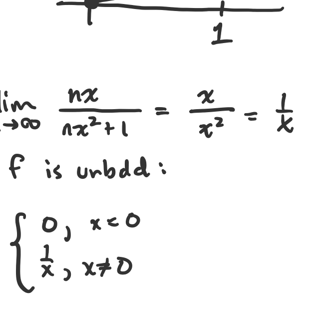
$$m < (m+n)\epsilon$$

$$(1-\epsilon)m < n$$

But this can't hold for all $m \in \mathbb{Z}^+$ since the left side grows to ∞ as $m \to \infty$ and will violate the inequality.

#3 | Find sequence $f_k : [0,1] \to \mathbb{R}$ ctn but converge ptwise to $f : [0,1] \to \mathbb{R}$ that is not bdd

Let $f_n(x) = \frac{nx}{nx^2 + 1}$



For all n, $f_n(0) = 0$.
For any $x \in (0,1]$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{nx}{nx^2+1} = \frac{x}{x^2} = \frac{1}{x}$$

Therefore, the function f is unbdd:

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \neq 0 \end{cases}$$