HW5 MTH 650 Fall 2025

Tuesday, October 14, 2025 11:49 AM

§2D #5

5 Prove that if $A \subseteq \mathbf{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subseteq F_2 \subseteq \cdots$ of closed sets contained in A such that

$$\left|A \setminus \bigcup_{k=1}^{\infty} F_k\right| = 0.$$

<u>Proof</u>: By Thm 2.71 $\forall \epsilon > 0$ $\exists$ closed $F \subseteq A$ such that $|A \setminus F| < \epsilon$.

Let $\epsilon_n = \frac{1}{n}$, then $\exists$ closed $F_n \subseteq A$ s.t. $|A \setminus F_n| < \frac{1}{n}$.

Let $F = cl\left(\bigcup_{n=1}^{\infty} F_n\right)$. Then

$$|A \setminus F| \underset{\boxed{X \subseteq cl(X)}}{\leq} \left|A \setminus \bigcup_{n=1}^{\infty} F_n\right| \underset{\boxed{\text{by construction}}}{<} \frac{1}{m} \text{ for all } m \in \{1,2,3,\dots\}$$

recall $cl(X)$ means closure of X, i.e.

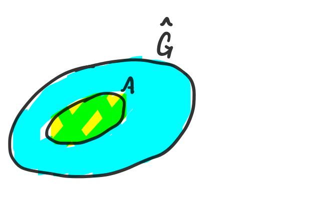
$cl(X) = X \cup \{y : y \text{ is a limit pt of } X\}$

$\sim cl(X)$ is always closed!

Therefore, F is closed and $|A \setminus F| = 0$. ∎

#6

6 Suppose $A \subseteq \mathbf{R}$ and $|A| < \infty$. Prove that A is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exists a set G that is the union of finitely many disjoint bounded open intervals such that $|A \setminus G| + |G \setminus A| < \varepsilon$.



<u>Proof</u>: Spz $A \subseteq \mathbb{R}$ and $|A| < \infty$.

($\rightarrow$) Spz A is Lebesgue msbl and let $\epsilon > 0$. By Thm 2.71 (e) $\exists$ open set $\hat{G}$ s.t. $A \subseteq \hat{G}$ and $|\hat{G} \setminus A| < \frac{\epsilon}{2}$. Since $\hat{G}$ is open, $\hat{G}$ can be written as a union of disjoint open intervals $\sim \hat{G} = \bigcup_{k=1}^{\infty} I_k$. ← WLOG assume $|I_{k+1}| \geq |I_k|$ for $k = 1, 2, \dots$ can always guarantee this by relabeling

Notice that $|\hat{G}| = |A \cup (\hat{G} \setminus A)|$ (disjoint) $= \underbrace{|A|}_{<\infty} + \underbrace{|\hat{G} \setminus A|}_{<\frac{\epsilon}{2}} < \infty$.

Thus,

$$|\hat{G}| = \left|\bigcup_{k=1}^{\infty} I_k\right| \underset{\{I_k\} \text{ are disjoint}}{=} \sum_{k=1}^{\infty} |I_k| < \infty$$

So series $\sum_{k=1}^{\infty} |I_k|$ converges, meaning $\forall \tilde{\epsilon} > 0\, \exists N\, \forall n > N$

$$\left|\sum_{k=1}^{\infty} |I_k| - \sum_{k=1}^{n} |I_k|\right| < \tilde{\epsilon}$$

positive, so can drop abs value

$$\underbrace{\sum_{k=1}^{\infty} |I_k| - \sum_{k=1}^{n} |I_k|}_{= \sum_{k=n+1}^{\infty} |I_k|} < \frac{\epsilon}{2}$$

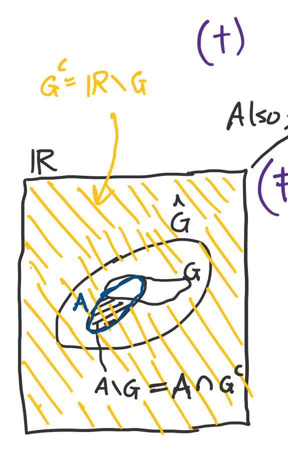
choose $\tilde{\epsilon} = \frac{\epsilon}{2}$ ← means $G = \bigcup_{k=1}^{n} I_k$ is $\frac{\epsilon}{2}$ away from $\hat{G}$ in terms of measure

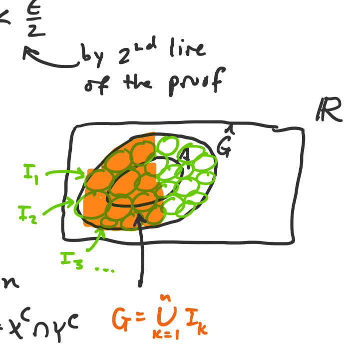
Let $G = \bigcup_{k=1}^{n} I_k$, a disjoint union of open intervals. Then, $G = \bigcup_{k=1}^{n} I_k$

$$|\hat{G} \setminus G| = |\hat{G}| - |G|$$
$$= \sum_{k=1}^{\infty} |I_k| - \sum_{k=1}^{n} |I_k| \qquad (\star)$$
$$= \sum_{k=n+1}^{\infty} |I_k| < \frac{\epsilon}{2}$$

Now since $G \subseteq \hat{G}$, $|G| \leq |\hat{G}|$. Thus,

$(\dagger)$ $\quad |G \setminus A| = |G| - |A| \leq |\hat{G}| - |A| = |\hat{G} \setminus A| < \frac{\epsilon}{2}$ ← by 2nd line of the proof





Also,

$(\ddagger)$ $|A \setminus G| = |A \cap G^c|$

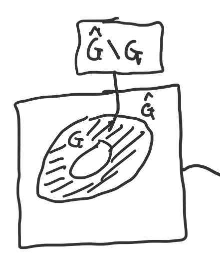
$= |[(A \cap G^c) \cap \hat{G}] \cup [(A \cap G^c) \cap \hat{G}^c]|$ (disjoint)

$= |A \cap G^c \cap \hat{G}| + \underbrace{|A \cap G^c \cap \hat{G}^c|}$ deMorgan $(X \cup Y)^c = X^c \cap Y^c$

$= A \cap (G \cup \hat{G})^c$

$= \phi$ ← b/c $G \cup \hat{G} = \hat{G}$ and $A \subseteq \hat{G}$

$\Rightarrow |A \cap G^c \cap \hat{G}^c| = 0$



$= |A \cap G^c \cap \hat{G}|$

$\leq |G^c \cap \hat{G}|$

$= |\hat{G} \setminus G|$

$\overset{(\star)}{<} \frac{\epsilon}{2}$.

Therefore, G is union of finitely many disjoint bdd open intervals and $(\dagger)$ and $(\ddagger)$,

$$|A \setminus G| + |G \setminus A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as was to be shown. ∎

($\leftarrow$) Spz $\forall \epsilon > 0$ $\exists \hat{G}$ s.t. $|A \setminus \hat{G}| + |\hat{G} \setminus A| < \epsilon$.

↑ union of finitely many disjoint open intervals

Let $\epsilon > 0$, so $\exists$ such a $\hat{G}$ s.t. $|A \setminus \hat{G}| + |\hat{G} \setminus A| < \frac{\epsilon}{2}$, so $\begin{cases} |\hat{G} \setminus A| < \frac{\epsilon}{2} - |\hat{G} \setminus A| \leq \frac{\epsilon}{2} \\ |A \setminus \hat{G}| < \frac{\epsilon}{2} - |\hat{G} \setminus A| \leq \frac{\epsilon}{2} \end{cases}$ $(\star)$

By def 2.2, $|A \setminus \hat{G}| = \inf_{\{I_k\}} \left\{\sum \ell(I_k)\right\}$, so since $|A \setminus \hat{G}| \leq \frac{\epsilon}{2}$, $\exists$ a particular

($\{I_k\}$ intervals, $A \setminus \hat{G} \subseteq \bigcup I_k$)

collection of intervals $\{I_k\}$ so that $|A \setminus \hat{G}| \leq \sum_{k=1}^{\infty} \ell(I_k) < \frac{\epsilon}{2}$. $(\star\star)$

Let $J = \bigcup_{k=1}^{\infty} I_k$, a union of open intervals, hence $\hat{G} \cup J$ is also open.

Compute

$$|(\hat{G} \cup J) \setminus A| = |(\hat{G} \cup J) \cap A^c|$$

$= |(\hat{G} \cap A^c) \cup (J \cap A^c)|$

subadditivity $\hookrightarrow \leq |\hat{G} \cap A^c| + |J \cap A^c|$

$\boxed{J \cap A^c \subseteq J} \rightarrow \leq |\hat{G} \setminus A| + |J|$

$(\star\star\star)$: $\overset{(\star),(\star\star)}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

So, given any $\epsilon > 0$, we let $G = \hat{G} \cup J$, so it is open and

$$|G \setminus A| = |(\hat{G} \cup J) \setminus A| \overset{(\star\star\star)}{<} \epsilon$$

Thus by Thm 2.71 (a) and (e), A is Lebesgue measurable. ∎

#7

7 Prove that if $A \subseteq \mathbf{R}$ is Lebesgue measurable, then there exists a decreasing sequence $G_1 \supseteq G_2 \supseteq \cdots$ of open sets containing A such that

$$\left|\left(\bigcap_{k=1}^{\infty} G_k\right) \setminus A\right| = 0.$$

<u>Pf</u>: Spz A is Lebesgue msbl. Let $n \in \{1,2,3,\dots\}$. By Thm 2.71 (e), $\exists$ open set $\hat{G}_n$ so that $A \subseteq \hat{G}_n$ and $|\hat{G}_n \setminus A| < \frac{1}{n}$. Let $G_k = \bigcap_{n=1}^{k} \hat{G}_n$.

Then we get $G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$ since intersections shrink sets and for all $N = \{1, 2, \dots\}$, $G_N \subseteq \hat{G}_N$, so $G_N \setminus A \subseteq \hat{G}_N \setminus A$ and

$$|G_N \setminus A| \leq |\hat{G}_N \setminus A| < \frac{1}{N}$$

Thus, as $N \to \infty$ we get

$$\left|\left(\bigcap_{k=1}^{\infty} \hat{G}_k\right) \setminus A\right| = 0. \; ∎$$

#12

12 Suppose $b < c$ and $A \subseteq (b,c)$. Prove that A is Lebesgue measurable if and only if $|A| + |(b,c) \setminus A| = c - b$.

<u>Proof</u>: ($\rightarrow$) Assume A is measurable, then we know $(b,c) \setminus A$ and A are disjoint and measurable, so

$$c - b \underset{\text{Thm 2.14}}{=} |(b,c)| \underset{\text{simple set thy}}{=} |[(b,c) \setminus A] \cup [A]| \underset{\text{measure property}}{=} |(b,c) \setminus A| + |A|,$$

(disjoint)

Completing proof in this direction. ∎

($\leftarrow$) Assume $A \subseteq (b,c)$ and

$$|A| + |(b,c) \setminus A| = c - b.$$

NTS A is Leb. msbl.

<u>Claim</u> For any $X \subseteq \mathbb{R}$ $\exists$ Borel set $B \supseteq X$ s.t. $|X| = |B|$.

<u>Proof</u>: If $|X| = \infty$, then let $B = \mathbb{R}$.

If $|X| < \infty$, then it means

$$|X| = \inf_{\substack{\{I_k\} \\ X \subset \bigcup I_k}} \left\{\sum \ell(I_k)\right\}$$

So let $\{I_{k,n}\}$ be intervals so that $X \subseteq \bigcup_{k=1}^{\infty} I_{k,n}$ and $\left|\bigcup_{k=1}^{\infty} I_{k,n}\right| = |X| + \frac{1}{n}$

Then the set

$$B = \bigcap_{n=1}^{\infty} \left[\bigcup_{k=1}^{\infty} I_{k,n}\right]$$

is a Borel set, $X \subseteq B$, and

since $\{I_{k,n}\}$ cover X for each n!

Borel set b/c it is made of unions and intervals of open intervals

$$|B| = |X| + \lim_{n \to \infty} \frac{1}{n} = |X|. \; ∎$$

By the claim, we can find a set X_1 so that $X_1 \supseteq A$, $|X_1| = |A|$ and X_1 is a Borel set and a set X_2 so that $X_2 \supseteq (b,c) \setminus A$, $|X_2| = |(b,c) \setminus A|$ and X_2 is a Borel set.

Since

$$|A| + |(b,c) \setminus A| = c - b,$$

we conclude

(i) $\quad |X_1| + |X_2| = c - b$.

Also

$$|X_2| = |(b,c) \setminus A| \underset{|M \setminus N| = |M| \setminus |N|}{=} |(b,c)| - |A| = c - b - |A|$$

So, we see that

$$|(b,c) \setminus X_2| = |(b,c)| - |X_2| = c - b - |X_2|$$
$$= \cancel{c} - b - [\cancel{c} - b - |A|]$$
$$= |A|$$

Therefore,

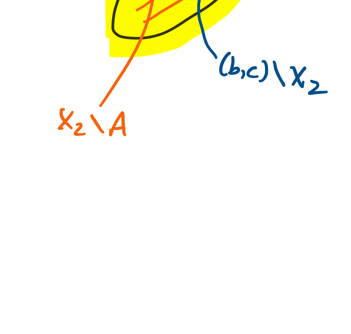
(ii) $\quad |X_2| + |(b,c) \setminus X_2| = |X_2| + (c - b - |X_2|) = c - b$

Thus by (i) and (ii),

$$|X_1| + |X_2| = c - b = |X_2| + |(b,c) \setminus X_2|$$
$$\Rightarrow |X_1| \overset{(\star)}{=} |(b,c) \setminus X_2|$$

And, since $A \subseteq (b,c)$, $A \setminus X_2 \subseteq (b,c) \setminus X_2 \subseteq A$, so

$$(b,c) \setminus X_2 \underset{\text{above line}}{\subseteq} A \underset{\text{def of } X_1}{\subseteq} X_1, \text{ hence}$$



$$|(b,c) \setminus X_2| \leq |A| \leq |X_1|.$$
$$\Rightarrow -A \overset{(\Delta)}{\leq} -|(b,c) \setminus X_2|$$

Now compute

$$|X_1 \setminus A| = |X_1| - |A|$$
$$\overset{(\Delta)}{=} |X_1| - |(b,c) \setminus X_2|$$
$$\overset{(\star)}{=} 0$$

Therefore we have found a Borel set $X_1 \supseteq A$ so that $|X_1 \setminus A| = 0$, and by Thm 2.71 we conclude that A is Lebesgue msbl. ∎