HW4 MTH 650 Fall 2025

Tuesday, October 14, 2025 10:25 AM

§2B #18 Suppose $f:\mathbb{R}\to\mathbb{R}$ is diff'bl on $\mathbb{R}$. Prove $f':\mathbb{R}\to\mathbb{R}$ is Borel-msbl.

<u>Proof</u>: Recall that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Let t_k be a sequence that converges to zero.

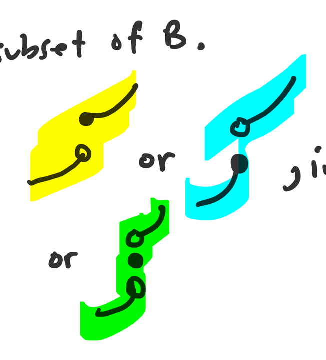
Then we can write

$$f'(x)=\lim_{k\to\infty}\frac{f(x+t_k)-f(x)}{t_k}$$

But since f is diff'bl, it is also continuous, hence by Thm 2.41, f is Borel-msbl. Also we can conclude for all k that $\frac{f(x+t_k)-f(x)}{t_k}$ is continuous, hence also Borel-msbl.

Thus by Thm 2.48, we see that f' is Borel-msbl since f' is a pointwise limit of Borel-msbl functions. ∎

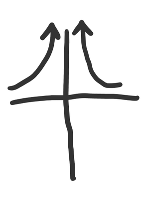
#22 Spz $B\subseteq\mathbb{R}$ and $f:B\to\mathbb{R}$ is increasing function. Prove f is continuous at all points of B except for a countable subset of B.

<u>Proof</u>: Since f is increasing, any discontinuity at $x=b$ must be of form  or , i.e. either

(i) $\lim_{x\to b^-} f(x)\neq f(b)$, or

(ii) $\lim_{x\to b^+} f(x)\neq f(b)$, or

(iii) $\lim_{x\to b^+} f(x)\neq f(b)$ AND $\lim_{x\to b^-} f(x)\neq f(b)$ ⟵ covered by cases (i) and (ii), so we won't need it

(<u>note</u>: asymptotes don't matter here because such points like $x=0$ in $\frac{1}{x^2}$ ⟶  are not in the domain and hence not important to us)

<u>Case (i)</u>: Here since f is increasing, $\lim_{x\to b^-} f(x)<f(b)$ so define the open interval $I_b=\left(\lim_{x\to b^-} f(x), f(b)\right)$

<u>Case (ii)</u>: Here $f(b)<\lim_{x\to b^+} f(x)$, so let $I_b=\left(f(b),\lim_{x\to b^+} f(x)\right)$.

Now let $b_1,b_2\in B$ with $b_1<b_2$. Assume f is discontinuous at both b_1 and b_2.

<u>Possibilities</u>

b_1	b_2	consequence
case (i)	case (i)	$\lim_{x\to b_1^-} f(x)<f(b_1)\le\lim_{x\to b_2^-} f(x)<f(b_2)\Rightarrow I_{b_1}\cap I_{b_2}=\emptyset$
(i)	(ii)	$\lim_{x\to b_1^-} f(x)<f(b_1)\le f(b_2)<\lim_{x\to b_2^+} f(x)\Rightarrow I_{b_1}\cap I_{b_2}=\emptyset$
(ii)	(i)	$f(b_1)<\lim_{x\to b_1^+} f(x)\le\lim_{x\to b_2^-} f(x)<f(b_2)\Rightarrow I_{b_1}\cap I_{b_2}=\emptyset$
(ii)	(ii)	$f(b_1)<\lim_{x\to b_1^+} f(x)\le f(b_2)<\lim_{x\to b_2^+} f(x)\Rightarrow I_{b_1}\cap I_{b_2}=\emptyset$

So in all cases, $I_{b_1}\cap I_{b_2}=\emptyset$.

But there must be at most countably many such I_b intervals because $I_b\cap\mathbb{Q}\neq\emptyset$ and so to each I_b we can pick a rational $q_b\in I_b\cap\mathbb{Q}$ and the function $g:\mathbb{R}\to\mathbb{Q}$ $g(b)=q_b$ shows there are at most countably many such I_b intervals (if uncountably many, then we would have uncountably many q_b's, which is not possible since $\mathbb{Q}$ is ctbl). ∎

#28 Spz $f:B\to\mathbb{R}$ is Borel-msbl. Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases} f(x), & x\in B\\ 0, & x\in\mathbb{R}\setminus B\end{cases}$$

Prove that g is Borel msbl.

<u>Proof</u>: First notice that $f^{-1}(\mathbb{R})=B$, so B is a Borel set since f is Borel-msbl, and consequently $\mathbb{R}\setminus B$ is Borel set.

Let $\tilde{B}\subset\mathbb{R}$ be a Borel set. If $0\notin\tilde{B}$, then $g^{-1}(\tilde{B})=f^{-1}(\tilde{B})$, which is a Borel set since f is Borel-msbl.

If $0\in\tilde{B}$, then

$$\begin{aligned} g^{-1}(\tilde{B}) &= g^{-1}(\{0\}\cup(\tilde{B}\setminus\{0\}))\\ &= g^{-1}(\{0\})\cup g^{-1}(\tilde{B}\setminus\{0\})\\ &= g^{-1}(\{0\})\cup \underbrace{f^{-1}(\tilde{B}\setminus\{0\})}_{\text{a Borel set since } f \text{ is Borel-msbl}}\end{aligned}$$

generally $h^{-1}(X\cup Y)=h^{-1}(X)\cup h^{-1}(Y)$

Now $g^{-1}(\{0\})=\underbrace{(g^{-1}(\{0\})\cap B)}_{\substack{\text{this is just}\\ f^{-1}(\{0\})\\ \text{so it is Borel-msbl}}}\cup(g^{-1}(\{0\})\cap(\mathbb{R}\setminus B))$

It suffices to show that $g^{-1}(\{0\})\cap(\mathbb{R}\setminus B)$ is Borel-msbl. But by def of g, we see that

$$g^{-1}(\{0\})\cap(\mathbb{R}\setminus B)=\mathbb{R}\setminus B,$$

which we argued earlier is a Borel set.

Thus for any Borel set $\tilde{B}$, $g^{-1}(\tilde{B})$ is Borel, so g is Borel-msbl. ∎

§2C

#1 Why does there not exist a measure space (X,S,μ) s.t. $\{\mu(E):E\in S\}=[0,1)$.

<u>Soln</u>: We know that for all $Y\in S$, $Y\subseteq X$ and by Thm 2.57

$$\mu(Y)\le\mu(X).$$

Suppose $\mu(X)=a\in\mathbb{R}$. We see that

$$\{\mu(E):E\in S\}=[0,a],$$

a closed set. But $[0,1)$ is not closed, so it could not be the same set as $\{\mu(E):E\in S\}$.

#5 Spz (X,S,μ) is measure space s.t. $\mu(X)<\infty$. Prove that if $\mathcal{A}$ is a set of disjoint sets in S s.t. $\mu(A)>0$ for all $A\in\mathcal{A}$, then $\mathcal{A}$ is a countable set.

<u>Proof</u>: Since $\bigcup\mathcal{A}\subseteq X$, we know by Thm 2.57 that

$$0<\mu(\bigcup\mathcal{A})=\sum_{A\in\mathcal{A}}\mu(A)\le\mu(X)<\infty.$$

Define $\mathcal{A}_n=\{A\in\mathcal{A}:\mu(A)>\frac{1}{n}\}$. We see that for each n, $\bigcup\mathcal{A}_n\subseteq\bigcup\mathcal{A}$.

Thus,

$$\infty>\mu(X)\ge\mu(\bigcup\mathcal{A})\ge\mu(\bigcup\mathcal{A}_n)>\sum_{A\in\mathcal{A}_n}\frac{1}{n}=\frac{1}{n}\operatorname{card}(\mathcal{A}_n).$$

Thus $\operatorname{card}(\mathcal{A}_n)$ must be finite (else the inequality above is violated).

Since $\mathcal{A}=\bigcup_{n=1}^{\infty}\mathcal{A}_n$, we see that $\mathcal{A}$ is a countable union of finite sets, which is countable, completing the proof. ∎

#9 Spz μ and ν are measures on msbl space (X,S). Prove that $\mu+\nu$ is also a measure on (X,S).

<u>Proof</u>: Suffices to show by def 2.54 that

(i) $(\mu+\nu)(\emptyset)=0$, and

(ii) for every disjoint sequence $E_1,\ldots\in S$, $(\mu+\nu)\left(\bigcup_{k=1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}(\mu+\nu)(E_k)$

To see (i), compute

$$(\mu+\nu)(\emptyset)\underset{\substack{\text{def of}\\ \mu+\nu}}{=}\mu(\emptyset)+\nu(\emptyset)\underset{\substack{\mu\text{ and}\\ \nu\text{ are}\\ \text{measures}}}{=}0+0=0$$

To see (ii),

$$(\mu+\nu)\Big(\underbrace{\dot{\bigcup_{k=1}^{\infty}}E_k}_{\substack{\text{union of}\\ \text{disjoint}\\ \text{sets}}}\Big)\underset{\substack{\text{def}\\ \text{of }\mu+\nu}}{=}\mu\left(\bigcup_{k=1}^{\infty}E_k\right)+\nu\left(\bigcup_{k=1}^{\infty}E_k\right)\underset{\substack{\mu\text{ and }\nu\\ \text{are measures}}}{=}\sum_{k=1}^{\infty}\mu(E_k)+\sum_{k=1}^{\infty}\nu(E_k)\underset{\substack{\text{def of}\\ \mu+\nu}}{=}\sum_{k=1}^{\infty}(\mu+\nu)(E_k)$$

Thus $\mu+\nu$ is a measure on (X,S). ∎

#10 Give example of msr space (X,S,μ) and decreasing sequence $E_1\supseteq E_2\supseteq\cdots$ of sets in S such that

$$\mu\left(\bigcap_{k=1}^{\infty}E_k\right)\neq\lim_{k\to\infty}\mu(E_k).$$

<u>Proof</u>: By Thm 2.60, we must assume $\mu(E_1)=\infty$ or else this will fail.

Consider $X=\mathbb{N}=\{1,2,\ldots\}$, $S=\mathcal{P}(X)$, $\mu=$ counting measure

Then let

$$\begin{aligned} E_1&=\mathbb{N}=\{1,2,\ldots\}\\ E_2&=\{2,3,\ldots\}\\ &\vdots\\ E_k&=\{k,k+1,\ldots\}\end{aligned}$$

Here we get $E_1\supseteq E_2\supseteq E_3\supseteq\cdots$

Each E_k has infinite measure:

$$\mu(E_k)=\operatorname{card}(E_k)=\infty$$

But $\bigcap_{k=1}^{\infty}E_k=\emptyset$, so

$$\mu\left(\bigcap_{k=1}^{\infty}E_k\right)=\mu(\emptyset)=0\neq\infty=\lim_{k\to\infty}\mu(E_k)$$