HW3 MTH 650 Fall 2025

Thursday, September 4, 2025 11:09 AM

$\emptyset \in S$; if $E \in S$, then $X \setminus E \in S$; $E_1, E_2, \ldots \in S \to \bigcup_{k=1}^{\infty} E_k \in S$

Section 2B

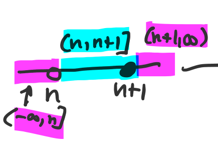
#1 Show $S = \left\{ \bigcup_{n \in K} (n, n+1] : K \subseteq \mathbb{Z} \right\}$ is a σ-algebra on $\mathbb{R}$.

Proof: Take $K_1 = \emptyset \subset \mathbb{Z}$. Then we have the empty union

$$\bigcup_{n \in K_1} (n, n+1] = \emptyset \in S.$$

Let $E \in S$, so $\exists K_2 \subset \mathbb{Z}$ s.t. $E = \bigcup_{n \in K} (n, n+1]$.

Then, $\mathbb{R} \setminus E = \bigcap_{n \in K} (\mathbb{R} \setminus (n, n+1])$



$$= \bigcap_{n \in K} \left[(-\infty, n] \cup (n+1, \infty) \right]$$

deMorgan law

$$= \left(\bigcap_{n \in K} (-\infty, n] \right) \cup \left(\bigcap_{n \in K} (n+1, \infty) \right) \quad (*)$$

If $\max K = \infty$, then $\bigcap_{n \in K} (n+1, \infty) = \emptyset$. ← call this $(\max K, \infty)$

If $\max K < \infty$, then $\bigcap_{n \in K} (\max K, \infty)$.

If $\min K = -\infty$, then $\bigcap_{n \in K} (-\infty, n] = \emptyset$ ← call this $(-\infty, \min K]$

If $\min K > -\infty$, then $\bigcap_{n \in K} (-\infty, n] = (-\infty, \min K]$.

So, $(*)$

$$\mathbb{R} \setminus E = (-\infty, \min K] \cup (\max K, \infty)$$

Therefore, taking

$$\tilde{K} = \{\ldots, -3, -2, -1, \ldots, -1 + \min(K), \max(K), 1 + \max(K), 2 + \max(K), \ldots\} \subseteq \mathbb{Z}$$

Gives

$$\mathbb{R} \setminus E = \bigcup_{n \in \tilde{K}} (n, n+1] = \left(\bigcup_{n=-\infty}^{\min(K)-1} (n, n+1] \right) \cup \left(\bigcup_{n = \max K}^{\infty} (n, n+1] \right)$$

$$= (-\infty, \min(K)] \cup (\max K, \infty) \in S.$$

Now suppose $E_1, E_2, \ldots \in S$, so $\exists K_1, K_2, \ldots \subseteq \mathbb{Z}$ s.t. for all $\ell = 1, 2, \ldots$

$$E_\ell = \bigcup_{n \in K_\ell} (n, n+1].$$

Let $K = \bigcup_{\ell=1}^{\infty} K_\ell \subseteq \mathbb{Z}$. So we have

$$\bigcup_{m=1}^{\infty} E_m = \bigcup_{m=1}^{\infty} \bigcup_{n \in K_m} (n, n+1]$$

$$= \bigcup_{\ell \in K} (n, n+1] \in S,$$

Thus S is a σ-algebra. ∎

#3 Suppose S is smallest σ-algebra on $\mathbb{R}$ containing $\{(r, s] : r, s \in \mathbb{Q}\}$. Prove that S is the collection of Borel subsets of $\mathbb{R}$.

Proof: Let $\mathcal{B} = \{X \subseteq \mathbb{R} : X \text{ is a Borel subset of } \mathbb{R}\}$.

Claim: $S \subseteq \mathcal{B}$

Pf: Let $(a, b] \in S$, so $\mathbb{R} \setminus (a, b] = (-\infty, a] \cup (b, \infty) \in S$. $(\dagger)$

But

$(*)$ $(-\infty, a] = \bigcap_{k=1}^{\infty} (-\infty, a + \frac{1}{k})$,

so each $(-\infty, a + \frac{1}{k}) \in \mathcal{B}$ since $\mathcal{B}$ contains all open intervals.

Since $\mathcal{B}$ is a σ-algebra, Theorem 2.25 applied to $(*)$ shows $(-\infty, a] \in \mathcal{B}$.

So we know $(-\infty, a] \in \mathcal{B}$ and $(b, \infty) \in \mathcal{B}$, thus $(\dagger)$ shows $\mathbb{R} \setminus (a, b] \in \mathcal{B}$.

Claim: $\mathcal{B} \subseteq S$

Pf:

Claim: $\underbrace{\{\text{open subsets of } \mathbb{R}\}}_{=:Y} \subseteq S$.

Pf: Let $J \subseteq \mathbb{R}$ be an open set. This means for all $x \in J$ $\exists \delta_x > 0$ s.t. $(x - \delta_x, x + \delta_x) \subset J$.

Thus

$$J = \bigcup_{x \in J} (x - \delta_x, x + \delta_x).$$

Write each

$$(x - \delta_x, x + \delta_x) = \bigcap_{k=1}^{\infty} \overbrace{(x - \delta_x, x + \delta_x + \tfrac{1}{k}]}^{\text{might not be of form } (r,s] \text{ for } r,s \in \mathbb{Q}}$$

For each x, if $x - \delta_x \in \mathbb{Q}$, then let $\ell_x = x - \delta_x$.

If $x - \delta_x \notin \mathbb{Q}$, then choose $\ell_x \in (x - \delta_x, x) \cap \mathbb{Q}$.

If $x + \delta_x + \frac{1}{k} \in \mathbb{Q}$, then choose $r_x = x + \delta_x + \frac{1}{k}$.

If $x + \delta_x + \frac{1}{k} \notin \mathbb{Q}$, then choose $r_x \in (x + \delta_x + \frac{1}{k+1}, x + \delta_x + \frac{1}{k}) \cap \mathbb{Q}$.

$+\frac{1}{k}) \cap \mathbb{Q}$

So we have

$$(x - \delta_x, x + \delta_x) = \bigcap_{k=1}^{\infty} (\ell_x, r_x]$$

which is an intersection of elements of form $(r, s]$ with $r, s \in \mathbb{Q}$.

Thus each $(x - \delta_x, x + \delta_x) \in S$ because S is a σ-algebra and Thm 2.25(c).

We conclude $J \in S$, since J is the union of countably many elements of S.

So we must conclude that $\mathcal{B} \subseteq S$ because, generally, if $X \subset Y$ then

"smallest σ-alg containing X" $\subset$ "smallest σ-alg cont. Y".

and in our case we have

$$\{\text{open subsets of } \mathbb{R}\} \subset \{(r, s] : r, s \in \mathbb{Q}\}$$

from which we conclude

$\mathcal{B}$ = smallest σ-alg cont $\{\text{open subsets of } \mathbb{R}\} \subset$ smallest σ-alg cont $\{(r, s] : r, s \in \mathbb{Q}\}$

#7 Prove if $B \subseteq \mathbb{R}$ is a Borel set and $t \in \mathbb{R}$, then $t + B$ is a Borel set.

Proof: Let $\mathcal{B}$ denote the collection of Borel sets.

Consider the translations

$$\mathcal{B}_t = \{X \in \mathcal{B} : t + X \in \mathcal{B}\}.$$

By def, $\mathcal{B}_t \subseteq \mathcal{B}$.

Is $\mathcal{B}_t$ a σ-algebra?

(1) $\emptyset \in \mathcal{B}_t$ because $\emptyset \in \mathcal{B}$ and $t + \emptyset = \emptyset$

(2) If $X \in \mathcal{B}_t$, then $\exists Y \in \mathcal{B}$ s.t. $X = t + Y$.

We know $\mathbb{R} \setminus Y \in \mathcal{B}$, thus $t + \mathbb{R} \setminus Y \in \mathcal{B}_t$. NTS $\mathbb{R} \setminus (t + Y) \in \mathcal{B}_t$.

Claim: $\mathbb{R} \setminus (t + Y) = t + \mathbb{R} \setminus Y$

Pf: Let $w \in \mathbb{R} \setminus (t + Y)$, so $w \notin t + Y$,

so $-t + w \notin -t + (t + Y) = Y$.

Thus, $-t + w \in \mathbb{R} \setminus Y$ and

$t + (-t + w) = w \in t + \mathbb{R} \setminus Y$.

$(\dagger)$ Thus $\mathbb{R} \setminus (t + Y) \subseteq t + \mathbb{R} \setminus Y$.

Now suppose that

$w \in t + \mathbb{R} \setminus Y$, so $\exists u \in \mathbb{R} \setminus Y$ s.t. $w = u + t$.

So, $u \notin Y$, thus $w = u + t \notin t + Y$.

Therefore, $w \in \mathbb{R} \setminus (t - Y)$.

We have shown

$(\ddagger)$ $t + \mathbb{R} \setminus Y \subseteq \mathbb{R} \setminus (t + Y)$.

The claim is proved by $(\dagger)$ and $(\ddagger)$.

Since $t + \mathbb{R} \setminus Y \in \mathcal{B}_t$, the claim shows $\mathbb{R} \setminus (t + Y) \in \mathcal{B}_t$.

(3) Let $E_1, E_2, \ldots \in \mathcal{B}_t$. Then $\exists F_1, F_2, \ldots \in \mathcal{B}$ so that each

$$E_k = t + F_k.$$

We know $\bigcup_{k=1}^{\infty} F_k \in \mathcal{B}$ because $\mathcal{B}$ is a σ-algebra.

NTS: $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} t + F_k \in \mathcal{B}_t$.

Claim: $\bigcup_{k=1}^{\infty} t + F_k = t + \bigcup_{k=1}^{\infty} F_k$.

Pf: Let $x \in \bigcup_{k=1}^{\infty} t + F_k$, so $\exists j$ s.t. $x \in t + F_j$.

So, $x - t \in F_j$, hence $x - t \in \bigcup_{k=1}^{\infty} F_k$.

Therefore, $x = (x - t) + t \in t + \bigcup_{k=1}^{\infty} F_k$.

Thus $\bigcup_{k=1}^{\infty} t + F_k \subseteq t + \bigcup_{k=1}^{\infty} F_k$. $(\blacktriangle)$

Conversely, let $x \in t + \bigcup_{k=1}^{\infty} F_k$. So $x - t \in \bigcup_{k=1}^{\infty} F_k$.

Thus $\exists j$ s.t. $x - t \in F_j$, hence $x \in t + F_j$.

We conclude $x \in \bigcup_{k=1}^{\infty} t + F_k$.

Therefore, $t + \bigcup_{k=1}^{\infty} F_k \subseteq \bigcup_{k=1}^{\infty} t + F_k$. $(\blacktriangle\blacktriangle)$

Hence by $(\blacktriangle)$ and $(\blacktriangle\blacktriangle)$ prove the claim.

So we have that $\mathcal{B}_t$ is a σ-algebra by (1), (2), and (3).

Given any open interval $(a, b) \in \mathcal{B}$, we also know $(a - t, b - t) \in \mathcal{B}$.

Therefore

$$t + (a - t, b - t) = (a, b) \in \mathcal{B}_t.$$

Thus $\mathcal{B}_t$ contains all open intervals and is a σ-algebra, so it must contain $\mathcal{B}$, i.e. $\mathcal{B}_t \subseteq \mathcal{B}$. ∎

#13 Spz (X, S) is a msbl space, $E_1, \ldots, E_n$ are mutually disjoint subsets of X (i.e. $E_i \cap E_j = \emptyset$ whenever $i \neq j$), and $c_1, \ldots, c_n \in \mathbb{R}$ are distinct real numbers.

Prove that $c_1 \chi_{E_1} + c_2 \chi_{E_2} + \cdots + c_n \chi_{E_n}$ is an S-msbl function iff $E_1, \ldots, E_n \in S$.

Proof: First suppose $n = 1$. We now argue $c_1 \chi_{E_1}$ is an S-msbl function iff $E_1 \in S$.

Let B be a Borel set and let $E_1 \subset X$.

By Def 2.37

$$c_1 \chi_{E_1}(t) = \begin{cases} E_1, & 0 \notin B \wedge c_1 \in B \\ X \setminus E_1, & 0 \in B \wedge c_1 \notin B \\ X, & 0 \in B \wedge c_1 \in B \\ \emptyset, & 0 \notin B \wedge c_1 \notin B \end{cases}$$

$(\rightarrow)$ Suppose $c_1 \chi_{E_1}$ is S-msbl, i.e. that for all Borel sets B, $(c_1 \chi_{E_1})^{-1}(B) \in S$.

Case 1: $(0 \notin B \wedge c_1 \in B)$ In this case,

$$(c_1 \chi_{E_1})^{-1}(B) = E_1 \in S$$

by hypothesis

In this case, we conclude $E_1 \in S$, as needed.

Case 2: $(0 \in B \wedge c_1 \notin B)$ Here,

$$(c_1 \chi_{E_1})^{-1}(B) = X \setminus E_1 \in S.$$

Since S is a σ-algebra,

$$E_1 = X \setminus (X \setminus E_1) \in S,$$

as needed.

Case 3: $(0 \in B \wedge c_1 \in B)$ Here,

$$(c_1 \chi_{E_1})^{-1}(B) = X \in S$$

since S is a σ-algebra.

Case 4: $(0 \notin B \wedge c_1 \notin B)$ Here,

$$(c_1 \chi_{E_1})^{-1}(B) = \emptyset \in S$$

since S is a σ-algebra.

Since we checked all cases, we have completed the proof for $n = 1$ in this direction.

$(\leftarrow)$ Suppose $E_1 \in S$. We NTS $c_1 \chi_{E_1}$ is S-msbl.

So let B be a Borel set.

We again have

Case	$(c_1 \chi_{E_1})^{-1}(B)$
1	$E_1 \in S$ by hypothesis
2	$X \setminus E_1 \in S$ since S is σ-alg.
3	$X \in S$ since S is σ-alg.
4	$\emptyset \in S$ since S is σ-alg.

Thus we showed $c_1 \chi_{E_1}$ is S-msbl.

So we have shown the result is true with $n = 1$.

Now suppose it holds for $n = N$, i.e.

$$c_1 \chi_{E_1} + c_2 \chi_{E_2} + \cdots + c_N \chi_{E_N} \text{ is } S\text{-msbl}$$

iff

$$E_1, E_2, \ldots, E_N \in S.$$

NTS $c_1 \chi_{E_1} + \cdots + c_N \chi_{E_N} + c_{N+1} \chi_{E_{N+1}}$ is S-msbl

iff

$$E_1, \ldots, E_{N+1} \in S.$$

$(\rightarrow)$ Suppose $c_1 \chi_{E_1} + \cdots + c_{N+1} \chi_{E_{N+1}}$ is S-msbl, i.e. for all Borel sets B,

$$(c_1 \chi_{E_1} + \cdots + c_{N+1} \chi_{E_{N+1}})^{-1}(B) \in S.$$

If $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{N+1} \notin B$ and $c_j \in B$, then

$$(c_1 \chi_{E_1} + \cdots + c_{N+1} \chi_{E_{N+1}})^{-1}(B) = E_j \in S$$

by hypothesis. This is true for each $j = 1, 2, \ldots, N+1$, completing the proof in this direction.

$(\leftarrow)$ We know that by the inductive hypothesis that

$$c_1 \chi_{E_1} + \cdots + c_N \chi_{E_N} \text{ is } S\text{-msbl}$$

and

$$c_{N+1} \chi_{E_{N+1}} \text{ is } S\text{-msbl}.$$

By Thm 2.46(a), we can conclude that the sum of these S-msbl functions is S-msbl, i.e.

$$c_1 \chi_{E_1} + \cdots + c_{N+1} \chi_{E_{N+1}} \text{ is } S\text{-msbl}.$$

This completes the proof. ∎

#17 Spz X is a Borel msbl subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$ is a fnct. st $W := \{x \in X : f \text{ is not continuous at } x\}$ is a countable set.

Prove that f is Borel-msbl function.

Proof: Let $c_1, c_2, \ldots \in W_x$ be the elements of W_x.

NTS that for all Borel sets B, $f^{-1}(B) \in \mathcal{B}$.

a Borel set

collection of all all Borel sets

By Thm 2.39, it is sufficient to argue that for all $a \in \mathbb{R}$,

$$f^{-1}((a, \infty)) \in \mathcal{B}.$$

So let $a \in \mathbb{R}$. Then

$$f^{-1}((a, \infty)) = (f^{-1}((a, \infty)) \cap W) \cup (f^{-1}((a, \infty)) \cap (X \setminus W)).$$

f ctn iff $\forall$ open sets θ, $f^{-1}(\theta)$ is open set

Since f is continuous on $X \setminus W$, we know that

$$f^{-1}((a, \infty)) \cap (X \setminus W)$$

is a (relatively) open subset of X.

We say A is rel. open in X if $A = X \cap \{\text{an open set}\}$

Thus $\exists$ open set U so that

a Borel set since U is open

$$f^{-1}((a, \infty)) \cap (X \setminus W) = U \cap (X \setminus W).$$

a Borel set b/c example 2.30 shows ctbl sets are Borel combined w/ fact that complements of Borel sets are Borel

Since intersections of Borel sets are Borel sets,

$$f^{-1}((a, \infty)) \cap (X \setminus W)$$

is a Borel set.

But also $f^{-1}((a, \infty)) \cap W$ is a countable set, so it is Borel set by Example 2.30.

Thus we have expressed $f^{-1}((a, \infty))$ as the union of Borel sets, so $f^{-1}((a, \infty))$ is a Borel set, since $\mathcal{B}$ is a σ-algebra. ∎