HW2 MTH 650 Fall 2025

Thursday, September 4, 2025 11:09 AM

§2A

#1 Proof: Spz $A, B \subseteq \mathbb{R}$ and $|B| = 0$.

NTS: $|A \cup B| = |A|$

Since $A \subseteq A \cup B$, Thm 2.5 shows

(i) $|A| \leq |A \cup B|$.

By Thm 2.8,

(ii) $|A \cup B| \leq |A| + |B|$

Therefore,

$$|A| \overset{(i)}{\leq} |A \cup B| \overset{(ii)}{\leq} |A| + \underbrace{|B|}_{=0 \text{ by hypothesis}} = |A|.$$

Therefore,

$$|A \cup B| = |A|. \blacksquare$$

#2 Let $A \subseteq \mathbb{R}, t \in \mathbb{R}$. Define

$$tA := \{ta : a \in A\}.$$

Prove $|tA| = |t|\,|A|$ (assuming $0 \cdot \infty := 0$)

Proof: First suppose that $t = 0$.

Then $tA = \{0\}$, which is a countable set,

so by Thm 2.4 we get here that $|0A| = 0$.

Thus $|0A| = 0 = |0|\,|A|$ doesn't matter if $|A|$ finite or ∞!

Suppose now $w > 0$.

If $I = (a,b)$ is an interval, then $wI = (wa, wb)$.

We have $\ell(I) = b - a$ and $\ell(wI) = w(b-a) = w\ell(I) = |w|\ell(I)$.

If $s < 0$, then $|sI| = (sb, sa)$ and $|sI| = sa - sb = s(a-b)$

$= -s(b-a)$

$= |s|\ell(I)$

So we have for any $t \in \mathbb{R} \setminus \{0\}$ that

(i) $\ell(tI) = |tI| = |t|\ell(I) = |t|\,|I| = |t|\ell(I)$

Let $t \in \mathbb{R} \setminus \{0\}$.

Let $I_1, I_2, \dots$ be open sets so that $A \subset \bigcup_{k=1}^{\infty} I_k$.

Then, $tA \subset \bigcup_{k=1}^{\infty} tI_k$.

Therefore by def of $|tA|$ (def 2.2)

$$|tA| \leq \sum_{k=1}^{\infty} \ell(tI_k) \overset{(i)}{=} \sum_{k=1}^{\infty} |t|\ell(I_k)$$

$$= |t| \sum_{k=1}^{\infty} \ell(I_k)$$

Thus for any $t \in \mathbb{R} \setminus \{0\}$ and choice of $\{I_k\}$,

$$\frac{|tA|}{|t|} \leq \sum_{k=1}^{\infty} \ell(I_k)$$

Taking inf over all choices of $\{I_k\}$ then gives

$$\frac{|tA|}{|t|} \leq \inf_{\{I_k\}} \sum_{k=1}^{\infty} \ell(I_k) =: |A|$$

Thus, for all $t \in \mathbb{R} \setminus \{0\}$

$$\frac{|tA|}{|t|} \leq |A| \implies |tA| \overset{(*)}{\leq} |t|\,|A|.$$

$A \subset \mathbb{R}$ was arbitrary! Any set ok!

Now given any $t \in \mathbb{R} \setminus \{0\}$,

$$|A| = \left|\frac{tA}{t}\right| = \left|\tfrac{1}{t} tA\right| \overset{(*)}{\leq} \left|\tfrac{1}{t}\right| |tA|$$

Therefore,

(**) $|t|\,|A| \leq |tA|$

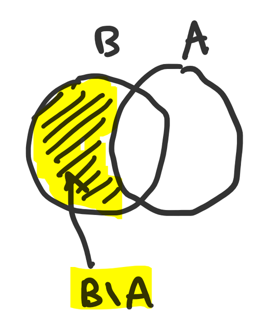
Thus by (*) and (**) we conclude

$$|t|\,|A| = |tA|,$$

completing the proof. $\blacksquare$

#3 Prove if $A, B \subseteq \mathbb{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

Proof: Notice that $A \cup B = A \cup (B \setminus A)$.



Since

$$B \subseteq B \cup A = A \cup (B \setminus A)$$

we see by Thm 2.5 that

$$|B| \leq |A \cup (B \setminus A)|$$

and by Thm 2.8

$$|B| \leq |A \cup (B \setminus A)| \leq |A| + |B \setminus A|$$

Therefore

$$|B \setminus A| \geq |B| - |A|. \blacksquare$$

#6 Prove that if $a, b \in \mathbb{R}$ with $a < b$, then

$$|(a,b)| = |[a,b)| = |(a,b]| = b - a$$

Proof: By Thm 2.14,

$$|[a,b]| = b - a.$$

By Thm 2.4, we know $|\{a,b\}| = 0$, $|\{b\}| = 0$, and $|\{a\}| = 0$.

By Problem 1,

$$|(a,b)| = |(a,b) \cup \{a,b\}| = |[a,b]| = b - a.$$

Similarly,

$$|[a,b)| = |[a,b) \cup \{b\}| = |[a,b]| = b - a$$

and

$$|(a,b]| = |(a,b] \cup \{a\}| = |[a,b]| = b - a,$$

completing the proof. $\blacksquare$

#8 Prove that if $A \subseteq \mathbb{R}$ and $t > 0$, then

$$|A| = |A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))|$$

Proof: Let $t > 0$. Clearly, since $(-t,t) \cup (\mathbb{R} \setminus (-t,t)) = \mathbb{R}$,

$$A = [A \cap (-t,t)] \cup [A \cap (\mathbb{R} \setminus (-t,t))]$$

Thus by Thm 2.8,

$$|A| = \left|[A \cap (-t,t)] \cup [A \cap (\mathbb{R} \setminus (-t,t))]\right|$$

(*)

$$\leq |A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))|.$$

If $|A| = \infty$, then we have

$$\infty \leq |A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))| \overset{\text{must be!}}{=} \infty$$

Completing the proof for $|A| = \infty$.

Let $\epsilon > 0$.

If $|A| < \infty$, then let $\{I_{1,k}\}$ and $\{I_{2,k}\}$ be collections of open intervals so that

$$|A \cap (-t,t)| \leq \frac{\epsilon}{2} + \sum_{k=1}^{\infty} I_{1,k} \quad \text{and}$$

$$|A \cap (\mathbb{R} \setminus (-t,t))| \leq \frac{\epsilon}{2} + \sum I_{2,k}$$

Then the collection $\{J_k\} = \{I_{m,k} : m = 1,2,\ k = 1,2,\dots\}$ has property that $A \subseteq \bigcup_{k=1}^{\infty} J_k$.

Also notice that

$$\sum_{k=1}^{\infty} \ell(J_k) = \sum_{k=1}^{\infty} (I_{1,k} + I_{2,k})$$

So,

$$|A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))| \leq \epsilon + \sum_{k=1}^{\infty} (\ell(I_{1,k}) + \ell(I_{2,k}))$$

$$= \epsilon + \sum_{k=1}^{\infty} \ell(J_k)$$

$$-\epsilon + |A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))| \leq \sum_{k=1}^{\infty} \ell(J_k)$$

Taking inf over all possible $\{J_k\}$ gives

$$|A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))| \leq \epsilon + |A|$$

Since ϵ was arbitrary, we get

(**) $|A \cap (-t,t)| + |A \cap (\mathbb{R} \setminus (-t,t))| \leq |A|$

Combine (*) and (**) to complete the proof. $\blacksquare$