HW1 MTH 650 Fall 2025

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§1A

#1 Proof: Let $P: x_0=a, x_1, x_2, \ldots, x_n=b$ be a partition and assume that

$$L(f,P,[a,b]) = U(f,P,[a,b]), \text{ i.e. that}$$

$$\sum_{k=1}^{n}(x_k-x_{k-1}) \inf_{[x_{k-1},x_k]} f = \sum_{k=1}^{n}(x_k-x_{k-1}) \sup_{[x_{k-1},x_k]} f.$$

Subtract to get

$$\sum_{k=1}^{n}(x_k-x_{k-1})\left[\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f\right] = 0$$

Each term of this sum is nonnegative since $x_k > x_{k-1}$ for all $k=1,\ldots,n$ and $\sup f \geq \inf f$ for all k.

Thus we must have

$$(\star) \qquad \sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f = 0 \quad \text{for all } k=1,\ldots,n,$$

otherwise the sum would be nonzero.

Since $\sup_{[x_{k-1},x_k]} f = \inf_{[x_{k-1},x_k]} f$ for all $k=1,2,\ldots,n$ we must conclude f is constant on each interval $[x_{k-1},x_k]$, because otherwise, $\exists\, t_1,t_2 \in [x_{k-1},x_k]$ so that $f(t_1) > f(t_2)$. But this would imply that $\sup_{[x_{k-1},x_k]} f > \inf_{[x_{k-1},x_k]} f$, contradicting $(\star)$.

To see f is constant on ALL of $[a,b]$, notice that the intervals of the partition

$$[x_0,x_1],[x_1,x_2],\ldots$$

all share an endpoint. The values of f must match at each x_i, else f would not be a function! So f is constant on $[a,b]$. This completes the proof. ■

#3 Proof: Let $f:[a,b]\to\mathbb{R}$ be bounded.

$(\rightarrow)$ Suppose f is Riemann integrable, i.e. that

$$U(f,[a,b]) = L(f,[a,b]), \text{ i.e.}$$

$$\inf_P U(f,P,[a,b]) = \sup_P L(f,P,[a,b])$$

Let $\epsilon>0$. By def of inf, $\exists$ partition P_1 so that

$$U(f,P_1,[a,b]) = U(f,[a,b]) + \tfrac{\epsilon}{2}$$

and by def of sup $\exists$ partition P_2 s.t.

$$L(f,P_2,[a,b]) = L(f,[a,b]) - \tfrac{\epsilon}{2}.$$

Let $P_3 = P_1 \cup P_2$. By Thm 1.5,

(i) $U(f,P_3,[a,b]) < U(f,P_1,[a,b]) = U(f,[a,b]) + \frac{\epsilon}{2}$

$L(f,P_2,[a,b]) > L(f,P_3,[a,b])$, so

(ii) $-L(f,P_3,[a,b]) < -L(f,P_2,[a,b]) = -L(f,[a,b]) + \frac{\epsilon}{2}$

Adding (i) and (ii) yields

$$U(f,P_3,[a,b]) - L(f,P_3,[a,b]) < U(f,[a,b]) - L(f,[a,b]) + \epsilon$$

But since f is Riemann-intg'bl, $U(f,[a,b]) - L(f,[a,b]) = 0$. Thus we have found partition P_3 s.t.

$$U(f,P_3,[a,b]) - L(f,P_3,[a,b]) < \epsilon, \text{ completing the proof. ■}$$

$(\leftarrow)$ Suppose $\forall \epsilon>0$ $\exists$ partition P_ϵ so that

$$U(f,P_\epsilon,[a,b]) - L(f,P_\epsilon,[a,b]) < \epsilon.$$

Suppose also for contradiction that f is not Riemann intg'bl, i.e. that $U(f,[a,b]) \neq L(f,[a,b])$. We can't have $L(f,[a,b]) < U(f,[a,b])$ by Thm 1.6, so we have

$$\inf_P U(f,P,[a,b]) \underset{\text{def}}{=} U(f,[a,b]) > L(f,[a,b]) \underset{\text{def}}{=} \sup_P L(f,P,[a,b])$$

Let $\epsilon^* = U(f,[a,b]) - L(f,[a,b]) > 0$. $(\star)$

By hypothesis, $\exists$ partition P_{ϵ^*} so that

$$(\star\star)\; U(f,P_{\epsilon^*},[a,b]) - L(f,P_{\epsilon^*},[a,b]) < \epsilon^*.$$

By inf and sup properties,

$$U(f,[a,b]) < U(f,P_{\epsilon}^*,[a,b])$$

and

$$L(f,[a,b]) > L(f,P_\epsilon^*,[a,b])$$

$$\Downarrow$$

$$-L(f,[a,b]) < -L(f,P_{\epsilon^*},[a,b])$$

Now, adding, we get

$$U(f,[a,b]) - L(f,[a,b]) < U(f,P_\epsilon^*,[a,b]) - L(f,P_\epsilon^*,[a,b]) \overset{(\star\star)}{<} \epsilon^*$$

But by $(\star)$, this means $\epsilon^* < \epsilon^*$, which isn't possible, a contradiction! This completes the proof. ■

#4 Proof: Let $f,g:[a,b]\to\mathbb{R}$ be Riemann intg'bl. By Problem 3, $\forall\epsilon>0$ $\exists$ partitions P_1,P_2 so that

$$(\dagger)\; U(f,P_1,[a,b]) - L(f,P_1,[a,b]) < \tfrac{\epsilon}{2}$$

and

$$(\ddagger)\; U(g,P_2,[a,b]) - L(g,P_2,[a,b]) < \tfrac{\epsilon}{2}.$$

Let $P^* = P_1 \cup P_2$. Then by Thm 1.5,

$$(\star)\left\{\begin{array}{l} U(f,P^*,[a,b]) \leq U(f,P_1,[a,b]) \\ L(f,P_1,[a,b]) \leq L(f,P^*,[a,b]) \\ \quad\Downarrow \\ -L(f,P^*,[a,b]) \leq -L(f,P_1,[a,b]) \end{array}\right\} \text{ and } (\star\star)\left\{\begin{array}{l} U(g,P^*,[a,b]) \leq U(g,P_2,[a,b]) \\ L(g,P_2,[a,b]) \leq L(g,P^*,[a,b]) \\ \quad\Downarrow \\ -L(g,P^*,[a,b]) \leq -L(g,P_2,[a,b]) \end{array}\right\}$$

Now, we need to show that $f+g$ is Riemann intg'bl, i.e.

$$\underbrace{U(f+g,[a,b])}_{\inf_P U(f+g,P,[a,b])\ (\text{def})} = \underbrace{L(f+g,[a,b])}_{\sup_P L(f+g,P,[a,b])\ (\text{def})}$$

To do this, adding in $(\star)$ and $(\star\star)$ with $(\dagger)$ and $(\ddagger)$ gives

(i) $U(f,P^*,[a,b]) - L(f,P^*,[a,b]) < \frac{\epsilon}{2}$ and

(ii) $U(g,P^*,[a,b]) - L(g,P^*,[a,b]) < \frac{\epsilon}{2}$.

By, definition,

finite sums!

$$U(f,P^*,[a,b]) + U(g,P^*,[a,b]) = \sum_k (x_k-x_{k-1})\sup_{[x_{k-1},x_k]} f + \sum_k (x_k-x_{k-1})\sup_{[x_{k-1},x_k]} g$$

$$= \sum_k (x_k-x_{k-1})\left[\sup_{[x_{k-1},x_k]} f + \sup_{[x_{k-1},x_k]} g\right]$$

recall $\sup(f+g) \leq \sup(f)+\sup(g)$

$$\geq \sum_k (x_k-x_{k-1}) \sup_{[x_{k-1},x_k]}(f+g)$$

$$= U(f+g,P^*,[a,b])$$

Similarly, we get

$$-\left[L(f,P^*,[a,b]) + L(g,P^*,[a,b])\right] \geq -L(f+g,P^*,[a,b])$$

Adding the previous two inequalities and using (i), (ii) gives

(iii) $U(f+g,P^*,[a,b]) - L(f+g,P^*,[a,b]) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, i.e. we have shown that $\forall\epsilon>0$ $\exists P^*$ s.t. (iii) holds.

By Problem 3, this implies $f+g$ is Riemann integrable. ■

#5 Proof: Let $f:[a,b]\to\mathbb{R}$ be Riemann intg'bl, so by Problem 3, $\forall\epsilon>0\;\exists P_\epsilon$ so that

$$U(f,P_\epsilon,[a,b]) - L(f,P_\epsilon,[a,b]) \overset{(\star)}{<} \epsilon.$$

Then,

$$U(f,P_\epsilon,[a,b]) = \sum_k (x_k-x_{k-1})\sup_{[x_{k-1},x_k]} f$$

recall $-\sup f = \inf(-f)$ $\Downarrow$ $\sup f = -\inf(-f)$

$$= -\sum_k (x_k-x_{k-1}) \inf_{[x_{k-1},x_k]} (-f)$$

$$= -L(-f,P_\epsilon,[a,b])$$

and similarly,

$$L(f,P_\epsilon,[a,b]) = -U(-f,P_\epsilon,[a,b]).$$

recall $-\inf x = \sup(-x)$ $\Downarrow$ $\inf x = -\sup(-x)$

Thus,

$$U(-f,P_\epsilon,[a,b]) - L(-f,P_\epsilon,[a,b]) = -L(f,P_\epsilon,[a,b]) + U(f,P_\epsilon,[a,b]) \overset{(\star)}{<} \epsilon,$$

and we see by Problem 3 that $U(-f,P_\epsilon) = L(-f,P_\epsilon)$, i.e. that $-f$ is Riemann intg'bl. ■

#11 Proof: Let $f:[a,b]\to\mathbb{R}$ be Riemann intg'bl and let

let's write "$\int_a^a f$" for this in the proof

$$F(t) = \begin{cases} 0, & t=a \\ \int_a^t f, & t\in(a,b] \end{cases}$$

$$\boxed{\begin{array}{l}\text{NTS } F \text{ is continuous on } [a,b], \text{ i.e.} \\ \forall t\in[a,b]\ \forall\epsilon>0\ \exists\delta>0 \text{ s.t. if } s\in(t-\delta,t+\delta)\cap[a,b] \\ \text{then } |F(t)-F(s)|<\epsilon.\end{array}}$$

thinking scratchwork

Need to get

$|F(t)-F(s)|<\epsilon$

$\left|\int_a^t f - \int_a^s f\right| < \epsilon$

$= \int_a^t f + \int_s^a f = \int_s^t f$

$\left|\int_s^t f\right| < \epsilon$

recall $|a|<b$ equivalent to $-b<a<b$

$-\epsilon < \int_s^t f < \epsilon$

problem 3 gives us partitions between!

$-\epsilon < -(U(f,P)-L(f,P)) < \int_s^t f < U(f,P)-L(f,P) < \epsilon$

or in other words,

$\underbrace{U(f,P)-L(f,P)}_{\text{always} > 0} < \epsilon$

$\left|\int_s^t f\right| = |U(f,[s,t])|$

Let $t\in[a,b]$ and $\epsilon>0$.

Since f is Riemann intg'bl on $[a,b]$, problem 9 shows it is also Riemann intg'bl on all $[c,d]\subset[a,b]$. Also, we know f is bounded, i.e. $\exists M>0$ s.t. $\forall x\in[a,b]$,

(a) $-M \leq f(x) \leq M$. ← hence $\sup_{[c,d]} f \leq M$

Choose $\delta < \frac{\epsilon}{M}$ $(\star)$. Let $s\in(t-\delta,t)$, i.e.

$$t-\delta < s < t$$
$$-\delta < s-t < 0$$
$$\text{(b) } t-s < \delta.$$

Let P be any partition of $[s,t]$. Then,

$$|F(t)-F(s)| = \left|\int_a^t f - \int_a^s f\right|$$

$$\overset{\text{Problem 10}}{=} \left|\cancel{\int_a^s f} + \int_s^t f - \cancel{\int_a^s f}\right|$$

$$= \left|\int_s^t f\right|$$

Either $\int_s^t f \geq 0$ or $\int_s^t f < 0$.

For now assume $\int_s^t f \geq 0$.

Then,

$$\left|\int_s^t f\right| = \int_s^t f$$

f is Riemann intg'bl

$$= U(f,[s,t])$$

inf property

$$\leq U(f,P,[s,t])$$

$$= \sum_k (x_k-x_{k-1}) \sup_{[x_{k-1},x_k]} f$$

$$\overset{(a)}{\leq} M\sum_k (x_k-x_{k-1})$$

(telescoping) sum of partition is length of interval $\ell([s,t])=t-s$

$$= M(t-s)$$

$$\overset{(b)}{<} \delta M$$

$$\overset{(\star)}{<} \left(\frac{\epsilon}{M}\right)M = \epsilon,$$

If $\int_s^t f < 0$, then

$$\left|\int_s^t f\right| = -\int_s^t f = -L(f,[s,t])$$

$L(f) = \sup_P L(f,P)$ → $L(f) \geq L(f,P)$ → $-L(f) \leq -L(f,P)$

$$\leq -L(f,P,[s,t])$$

$$= -\sum_k (x_k-x_{k-1}) \inf_{[x_{k-1},x_k]} f$$

$-\inf f = \sup(-f)$

$$= \sum_k (x_k-x_{k-1}) \sup_{[x_{k-1},x_k]} (-f)$$

$-M \leq f \leq M$ $\Downarrow$ $M \geq -f \geq -M$

$$\leq M \sum_k (x_k-x_{k-1})$$

$$= M(t-s)$$

$$< \delta M < \epsilon.$$

A similar proof works for $s\in(t,t+\delta)$.

This completes the proof. ■