This same reasoning shows that if Y and Z are figures, then Y and Z can be written as unions of intervals of the same grid Π. Therefore $Y \cup Z$ is a figure. Moreover,

$$V(Y \cup Z) \leq V(Y) + V(Z). \tag{5.2}$$

If $Y \cap Z$ is empty, then equality holds in (5.2).

PROBLEMS

1. Let $n = 1$ and $Y = [0, 1] \cup [2, 3]$, $Z = [1, 3] \cup [4, 5]$. Verify Formula (5.2) in this example.
2. Let $n = 2$ and $Y = [0, 2] \times [0, 1] \cup [1, 3] \times [1, 2]$, $Z = [-1, 2] \times [-1, 3]$. Find a grid Π such that both Y and Z are unions of intervals of Π. Find the areas of Y, Z, $Y \cup Z$, and $Y \cap Z$, and verify that $V(Y) + V(Z) - V(Y \cup Z) = V(Y \cap Z)$.
3. Let $I_1 = [0, 1] \times [0, 1] \times [0, 1]$ and $I_2 = [\frac{1}{2}, 2] \times [0, 2] \times [-1, 2]$. Find the volume of $I_1 \cup I_2$ and of $I_1 \cap I_2$.
4. Let m be a positive integer, $f(x) = \exp x$, and $Y = I_1 \cup \cdots \cup I_m$, where for $k = 1, \ldots, m$

$$I_k = [(k-1)/m, k/m] \times [0, f(k/m)].$$

   Find the area $V_2(Y)$. Show that it is approximately $e - 1$ if m is large.
5. (a) Let I_1 and I_2 be n-dimensional intervals. Show that $I_1 \cap I_2$ is also an interval provided that it has nonempty interior.
   (b) When is $I_1 \cup I_2$ an interval?

5.2 Measure

We now define measure in the Lebesgue sense for a large class of subsets of E^n. A set A in this class is called *measurable*, and the measure of A is denoted by $V(A)$. Some main properties of measurable sets and of measure are stated in Theorems 5.1 through 5.3. In studying the present section, the reader should first concentrate on understanding the definitions and the statements of these main theorems. A careful study of the various lemmas and propositions in the section might be postponed.

We begin by defining the measure of a bounded set A. This is done in two stages. First, the measure of an open set G is defined by approximating G from within by figures, and that of a compact set K by approximating K from without by figures. In the second stage, A is approximated from within by compact sets and from without by open sets. This two-stage approximation process is an important feature of the Lebesgue theory of measure.

There is an older theory of measure due to Jordan. In this theory A is approximated simultaneously from within and without by figures. The Jordan theory is unsatisfactory for several reasons. Among them is the fact

Hence A is a null set. Since $V_1(A) + V_1[(0, 1) - A] = V_1[(0, 1)] = 1$, the set of irrational numbers in $(0, 1)$ has measure 1 and therefore must be an uncountable set.

EXAMPLE 5. Let $A \subset M$, where M is an $(n - 1)$-manifold. It is plausible that the n-dimensional measure of A is 0, and this fact is proved in Section 5.8. Hence any such set A is a null set.

A sequence of sets $A_1, A_2, \ldots$ is called *monotone* if either $A_1 \subset A_2 \subset \cdots$ or $A_1 \supset A_2 \supset \cdots$. In the first instance the sequence is called *nondecreasing*, and in the second instance *nonincreasing*.

Theorem 5.3

(a) *Let $A_1, A_2, \ldots$ be a nondecreasing sequence of measurable sets. Then*

$$V\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) = \lim_{\nu\to\infty} V(A_\nu) \tag{5.11}$$

(b) *Let $A_1, A_2, \ldots$ be a nonincreasing sequence of measurable sets, such that $V(A_1) < \infty$. Then*

$$V\left(\bigcap_{\nu=1}^{\infty} A_\nu\right) = \lim_{\nu\to\infty} V(A_\nu) \tag{5.12}$$

PROOF. Let us prove the theorem under the assumption that there is a spherical ball U such that $A_\nu \subset U$ for each $\nu = 1, 2, \ldots$. This restriction will be removed in Section 5.10. To prove (a), define $B_1, B_2, \ldots$ as in the proofs of Theorems 5.1 and 5.2. Then

$$V\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) = \sum_{k=1}^{\infty} V(B_k).$$

Since $A_1 \subset A_2 \subset \cdots$, we have $B_1 \cup \cdots \cup B_\nu = A_\nu$. Therefore

$$V(A_\nu) = \sum_{k=1}^{\nu} V(B_k).$$

We get (5.11) by taking the limit as $\nu \to \infty$. To get (5.12), we apply (5.11) to the nondecreasing sequence of sets $C_\nu = U - A_\nu$, and note that $V(C_\nu) = V(U) - V(A_\nu)$ since $A_\nu \subset U$. □

EXAMPLE 6. To see the need for the assumption $V(A_1) < \infty$ in Theorem 5.3(b), let $n = 1$ and $A_\nu = [\nu, \infty)$. Then $A_1 \supset A_2 \supset \cdots$ and $V(A_\nu) = +\infty$ for each $\nu = 1, 2, \ldots$. However, $A_1 \cap A_2 \cap \cdots$ is empty, and hence $V(A_1 \cap A_2 \cap \cdots) = 0$.

Problems

In 1, 2, and 3 assume that the sets are bounded.

1. Let A and B be measurable. Show that
(a) $V(A - B) = V(A) - V(A \cap B)$.
(b) $V(A \cup B) + V(A \cap B) = V(A) + V(B)$.

2. Show that if A, B, and C are measurable, then

$$V(A \cup B \cup C) = V(A) + V(B) + V(C) - V(A \cap B) - V(A \cap C) - V(B \cap C) + V(A \cap B \cap C).$$

3. Show that if A is measurable and B is a null set, then

$$V(A \cup B) = V(A - B) = V(A).$$

4. Let $A = A_1 \cup A_2 \cup \cdots$, where $A_k = \{(x, y): x = 1/k, 0 \le y \le 1\}$ for $k = 1, 2, \ldots$ Show that $V_2(A) = 0$.

5. Let A_0 be the circular disk with center (0, 0) and radius 1. For $k = 1, 2, \ldots$, let A_k be the circular disk with center $(1 - 4^{-k})\mathbf{e}_1$ and radius 4^{-k-1}. Let $A = A_0 - (A_1 \cup A_2 \cup \cdots)$. Find $V_2(A)$.

6. Prove Lemma 1. [*Hint*: Consider the collection of all intervals I such that $I \subset G$. The interiors int I of these intervals form an open covering of K.]

7. Prove Corollary 1 to Proposition 5.1a. [*Hint*: If $K \subset G$, then $G = K \cup (G - K)$. By Proposition 5.1a, $V(G) = V(K) + V(G - K)$.]

8. (a) Show that if A and B are countable sets, then $A \cup B$ is countable.
(b) Show that if $B \subset A$ and A is countable, then B is countable.
(c) Show that if $A_1, A_2, \ldots$ are countable sets, then $A_1 \cup A_2 \cup \cdots$ is countable.

9. Let $A = \{x_1, x_2, \ldots\}$ be a countable subset of (0, 1). Given $0 < \varepsilon < 1$, let $\varepsilon_k = \varepsilon 2^{-k-1}$, $I_k = (x_k - \varepsilon_k, x_k + \varepsilon_k)$, and $G = I_1 \cup I_2 \cup \cdots$.
(a) Show that $V_1(G) \le \varepsilon$.
(b) In particular, let A be the set of rational numbers in (0, 1). Let $K = [0, 1] - G$. Then K is a compact subset of the irrational numbers. Show that $V_1(K) \ge 1 - \varepsilon$.
(c) Show that $K = \text{fr } K$.

10. Let C be the Cantor set, defined in Problem 5, Section 2.4.
(a) Show that C is a null set ($V_1(C) = 0$).
(b) Show that $x \in C$ if and only if $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i = 0$ or 2, $i = 1, 2, \ldots$.
(c) Let $f(x) = \sum_{i=1}^{\infty} a_i 2^{-i-1}$ for $x \in C$. Show that $f(C) = [0, 1]$. Hence C is uncountable.
(d) For x in the kth interval of A_j (Problem 5, Section 2.4) let $f(x)$ have the constant value $(2k - 1)2^{-j}$, $k = 1, 2, \ldots, 2^{j-1}$, $j = 1, 2, \ldots$. Show that f is continuous and nondecreasing on [0, 1]. [*Note*: f is called the *Cantor function*.]

11. Show that any straight line in E^2 has area 0.