give E^n a noneuclidean inner product. This affects the formula for changing covectors into vectors. (See the distinction between the differential and gradient of a function, Section 3.3.)

PROBLEMS

1. Let $n = 3$ and $L(x, y, z) = x + y + 2z$.
 (a) What is the covector $\mathbf{a} = (a_1, a_2, a_3)$ corresponding to L?
 (b) Describe the set $\{(x, y, z) : L(x, y, z) = c\}$ and the intersection of this set with the plane $\{(x, y, z) : y = x\}$.

2. Prove that any linear function L is continuous using Proposition 3.1 and the inequality $|\mathbf{a} \cdot \mathbf{z}| \leq |\mathbf{a}|\,|\mathbf{z}|$

3. Let L be linear on E^n.
 (a) Show that $\{\mathbf{x} : L(\mathbf{x}) = c\}$ is a hyperplane, unless $L(\mathbf{x}) = 0$ for all $\mathbf{x} \in E^n$.
 (b) Give another proof (not the one suggested in Problem 5, Section 1.4) that hyperplanes and closed half-spaces are closed sets, using Problem 2.

4. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for E^n. Define L by the formula

$$L(c^1\mathbf{x}_1 + \cdots + c^n\mathbf{x}_n) = c^n$$

for every $c^1, \ldots, c^n$. Show that:
 (a) L is a linear function.
 (b) The set $P = \{\mathbf{x} : L(\mathbf{x}) = 0\}$ is the $(n - 1)$-dimensional vector subspace spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$.

5. Let $\|\ \|$ be any norm on E^n (Section 2.11). Define on $(E^n)^*$ the *dual norm* as follows. For every covector $\mathbf{a}$,

$$\|\mathbf{a}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\}.$$

 (a) Verify that the dual norm satisfies Properties (1), (2), and (3) (Section 2.9).
 (b) Show that $\|\mathbf{x}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\}$. [*Hint*: Problem 7, Section 2.4.]

3.3 Differentiable functions

The existence of a derivative for a function of one variable is a fact of considerable interest. Geometrically, it says that a tangent line exists. However, the fact that a function of several variables has partial derivatives is not in itself of much interest. For one thing, the existence of derivatives in the directions of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ does not imply that derivatives exist in other directions. Moreover, the function need not have a tangent hyperplane even if there is a derivative in every direction (see Example 2 below).

We shall now define a more natural notion, that of differentiability. Geometrically, differentiability means the existence of a tangent hyperplane. It will be shown that most of the basic properties of differentiable functions of one variable remain true for differentiable functions of several variables.

Let us again consider an interior point $\mathbf{x}_0$ of the domain D of a real valued function f.

Note. This definition of the gradient vector is correct only if we use the euclidean inner product in E^n. Suppose that E^n is given some other inner product $B(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} c_{ij}x^i y^j$, with (c_{ij}) a symmetric, positive definite matrix (see Section 2.11). Let us denote the gradient vector with respect to this inner product by $\operatorname{grad}_B f(\mathbf{x})$. We require that $\operatorname{grad}_B f(\mathbf{x})$ satisfy

$$B(\operatorname{grad}_B f(\mathbf{x}), \mathbf{h}) = df(\mathbf{x}) \cdot \mathbf{h}, \quad \text{for all } \mathbf{h} \in E^n.$$

To find the components of the vector $\mathbf{z} = \operatorname{grad}_B f(\mathbf{x})$, let $\mathbf{h} = \mathbf{e}_i$ in (3.9). Then

$$\sum_{j=1}^{n} c_{ij} z^j = f_i(\mathbf{x}), \quad i = 1, \dots, n.$$

This is a system of linear equations for the components $z^1, \dots, z^n$ of $\operatorname{grad}_B f(\mathbf{x})$. Let (c^{ij}) denote the inverse of the matrix (c_{ij}). Then

$$z^i = \sum_{j=1}^{n} c^{ij} f_j(\mathbf{x}), \quad i = 1, \dots, n,$$

(3.12) $$\operatorname{grad}_B f(\mathbf{x}) = \sum_{i,j=1}^{n} c^{ij} f_j(\mathbf{x}) \mathbf{e}_i.$$

For a noneuclidean inner product, (3.12) replaces (3.11).

PROBLEMS

In Problems 1, 2, 3, and 9, assume that f is differentiable. In each case this follows from Theorem 3.2 in Section 3.4.

1. Let $f(x, y) = 3x^2y + 2xy^2$. Find the tangent plane at $(1, -2, 2)$.

2. Using the formula $df(\mathbf{x}_0) \cdot \mathbf{v}$ for directional derivative, find the derivative of f at $\mathbf{x}_0$ in the direction $\mathbf{v}$.
(a) $f(x, y) = xy$, $\mathbf{x}_0 = (1, 3)$, $\mathbf{v} = (2/\sqrt{5}, -1/\sqrt{5})$.
(b) $f(x, y) = x \exp(xy)$, $\mathbf{x}_0 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{v} = (1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$.
(c) $f(x, y, z) = ax^2 + by^2 + cz^2$, $\mathbf{x}_0 = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_3$.

3. Let $f(x, y) = \log(x^2 + 2y + 1) + \int_0^x \cos(t^2)dt$, $y > -\frac{1}{2}$.
(a) Find $df(x, y)$.
(b) Find approximately $f(0.03, 0.03)$.

4. Find $\operatorname{grad} f(\mathbf{x})$ for each of the following functions:
(a) $f(\mathbf{x}) = \mathbf{x}_0 \cdot \mathbf{x}$. (b) $f(\mathbf{x}) = |\mathbf{x}|$, $\mathbf{x} \neq \mathbf{0}$. (c) $f(\mathbf{x}) = (\mathbf{x}_0 \cdot \mathbf{x})^2$.

5. In Problem 3, Section 3.1, show that f is continuous at $(0, 0)$, but not differentiable at $(0, 0)$,

6. Let $f(x, y) = 2xy^2/(x^2 + y^4)$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$, as in Example 2.
(a) Show that $-1 \leq f(x, y) \leq 1$ for every (x, y).
(b) Find $\{(x, y): f(x, y) = 1\}$ and $\{(x, y): f(x, y) = -1\}$.
(c) Find $\{(x, y): \operatorname{grad} f(x, y) = (0, 0)\}$.
(d) Find $\{(x, y): f(x, y) = c\}$ for any c, and illustrate with a sketch.

7. Let f and g be differentiable at $\mathbf{x}_0$. (a) Prove that the sum $f + g$ is differentiable at $\mathbf{x}_0$, and $d(f + g)(\mathbf{x}_0) = df(\mathbf{x}_0) + dg(\mathbf{x}_0)$. (b) Prove that the product fg is differentiable at $\mathbf{x}_0$, and $d(fg)(\mathbf{x}_0) = f(\mathbf{x}_0)dg(\mathbf{x}_0) + g(\mathbf{x}_0)df(\mathbf{x}_0)$. [*Hint*: Recall the proof for $n = 1$.]

8. (*Euler's formula*.) Let p be a real number. A function f is called *homogeneous of degree* p if $f(t\mathbf{x}) = t^p f(\mathbf{x})$ for every $\mathbf{x} \neq \mathbf{0}$ and $t > 0$. Let f be differentiable for all $\mathbf{x} \neq \mathbf{0}$. Show that if f is homogeneous of degree p, then

$$df(\mathbf{x}) \cdot \mathbf{x} = pf(\mathbf{x})$$

for every $\mathbf{x} \neq \mathbf{0}$, and conversely. [*Hint*: Let $\phi(t) = f(t\mathbf{x})$ and use Proposition 3.2 with $\mathbf{x}_0 = \mathbf{0}$. For the converse, show that for fixed $\mathbf{x}$, $\phi(t)t^{-p}$ is a constant.]

9. Let $Q(\mathbf{x}) = \sum_{i,j=1}^{n} C_{ij}x^i x^j$, where $C_{ij} = C_{ji}$ and $Q(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{0}$. Let $f(\mathbf{x}) = [Q(\mathbf{x})]^{p/2}$. Calculate $df(\mathbf{x})$ and verify Euler's formula for this function.

10. Let f be continuous at $\mathbf{x}_0$ and g differentiable at $\mathbf{x}_0$ with $g(\mathbf{x}_0) = 0$. Show that the product fg is differentiable at $\mathbf{x}_0$.

3.4 Functions of class $C^{(q)}$

Let f be a function whose domain is an open set $D \subset E^n$.

Definition. If f is continuous on D, then f is said to be a *function of class* $C^{(0)}$. If the partial derivatives $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ exist for every $\mathbf{x} \in D$ and $f_1, \ldots, f_n$ are continuous functions on D, then f is a *function of class* $C^{(1)}$.

The classes $C^{(q)}$ of functions, where $q = 2, 3, \ldots$, are defined below. We first prove the following sufficient condition for differentiability, which is adequate for most purposes.

Theorem 3.2. *If f is a function of class $C^{(1)}$, then f is a differentiable function.*

Proof. Let us proceed by induction on the dimension n. If $n = 1$, differentiability means simply that $f'(x)$ exists for every $x \in D$, while if f is of class $C^{(1)}$, then f' is a continuous function. Let us assume that the theorem is true in dimension $n - 1$.

Let $\mathbf{x}_0$ be any point of D and $\delta_0 > 0$ such that the δ_0-neighborhood of $\mathbf{x}_0$ is contained in D. Let us write (Figure 3.3):

$$\hat{\mathbf{x}} = (x^1, \ldots, x^{n-1}), \qquad \hat{\mathbf{x}}_0 = (x_0^1, \ldots, x_0^{n-1}),$$
$$\phi(\hat{\mathbf{x}}) = f(x^1, \ldots, x^{n-1}, x_0^n) = f(\hat{\mathbf{x}}, x_0^n),$$

provided the point $(\hat{\mathbf{x}}, x_0^n)$ is in D. The partial derivatives of ϕ are

$$\phi_i(\hat{\mathbf{x}}) = f_i(\hat{\mathbf{x}}, x_0^n), \qquad i = 1, \ldots, n - 1.$$

Let us show that this function is of class $C^{(\infty)}$ and that $f^{(q)}(0) = 0$ for every $q = 1, 2, \ldots$. For $x \neq 0$ the derivatives $f^{(q)}(x)$ can be computed by elementary calculus, and each $f^{(q)}$ is continuous on $E^1 - \{0\}$. It is at the point 0 where f must be examined. Now

$$\lim_{u \to +\infty} u^k \exp(-u) = 0 \quad \text{for each } k = 0, 1, 2, \ldots, \tag{3.17}$$

a fact that we prove immediately below. If $x < 0$, then $f(x) = f'(x) = f''(x) = \cdots = 0$. Using (3.17) with $k = 0$, $\exp(-1/x^2) \to 0$ as $x \to 0^+$. Since $f(0) = 0$, f is continuous. If $x > 0$

$$f'(x) = \frac{2}{x^3} \exp\left(-\frac{1}{x^2}\right) = 2x \cdot \frac{1}{x^4} \exp\left(-\frac{1}{x^2}\right).$$

Using (3.17) with $k = 2$, $f'(x) \to 0$ as $x \to 0^+$. Therefore $\lim_{x\to 0} f'(x) = 0$. By Problem 4, $f'(0) = 0$ and f is of class $C^{(1)}$. For each $q = 2, 3, \ldots, f^{(q)}(x)$ is a polynomial in $1/x$ times $\exp(-1/x^2)$ for $x > 0$. Hence $\lim_{x \to 0} f^{(q)}(x) = 0$. By Problem 4 and induction on q, $f^{(q)}(0) = 0$ and $f \in C^{(q)}$ for every q. Thus $f \in C^{(\infty)}$. If we expand f by Taylor's formula about 0, then $f(x) = R_q(x)$ for every x. If $x > 0$ the remainder $R_q(x)$ does not tend to 0 as $q \to \infty$. Hence f is not an analytic function.

Proof of (3.17). For each $u < 0$ let $\psi(u) = u^{-k} \exp u$. Then

$$\psi'(u) = (u - k)u^{-k-1} \exp u,$$
$$\psi''(u) = [u^2 - 2ku + k(k + 1)]u^{-k-2} \exp u.$$

The expression in brackets has a minimum when $u = k$ and is positive there. Hence $\psi''(u) > 0$ for all $u > 0$. Let us apply Taylor's formula to ψ, with $q = 2$:

$$\psi(u) = \psi(u_0) + \psi'(u_0)(u - u_0) + \tfrac{1}{2}\psi''(v)(u - u_0)^2,$$

with v between u and u_0. Since $\psi''(v) > 0$,

$$\psi(u) \geq \psi(u_0) + \psi'(u_0)(u - u_0).$$

If $u_0 > k$, then $\psi'(u_0) > 0$ and the right-hand side tends to $+\infty$ as $u \to +\infty$. Hence $\psi(u) \to +\infty$ and $1/\psi(u) \to 0$ as $u \to +\infty$. □

Problems

1. Expand $f(x, y, z) = xyz$ by Taylor's formula about $\mathbf{x}_0 = (1, -1, 0)$, with $q = 4$.

2. Let $f(x, y) = x^{-1} \cos y$, $x > 0$, and $\mathbf{x}_0 = (1, 0)$.
   (a) Expand $f(x, y)$ by Taylor's formula about $\mathbf{x}_0$, with $q = 2$, and find an estimate for $|R_2(x, y)|$.
   (b) Show that $R_q(x, y) \to 0$ as $q \to \infty$ for (x, y) in some open set containing $\mathbf{x}_0$.

3. Let $f(x, y) = \psi(ax + by)$, where a and b are scalars and ψ is of class $C^{(q)}$ in some open set containing 0. Show that Taylor's formula about (0, 0) becomes

$$f(x, y) = \sum_{m=0}^{q-1} \frac{\psi^{(m)}(0)}{m!} \sum_{j=0}^{m} \binom{m}{j}(ax)^j(by)^{m-j} + R_q(x, y),$$

where $\binom{m}{j}$ is the binomial coefficient (which equals the number of j-element subsets of a set with m elements).

4. Let f be continuous on an open set D and of class $C^{(1)}$ on $D - \{\mathbf{x}_0\}$. Suppose, moreover, that $l_i = \lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x})$ exists for each $i = 1, \ldots, n$. Prove that $l_i = f_i(\mathbf{x}_0)$, and consequently that f is of class $C^{(1)}$ on D. State and prove a corresponding result in case $q > 1$. [*Hint*: Apply the mean value theorem to $f(\mathbf{x}_0 + t\mathbf{e}_i) - f(\mathbf{x}_0)$.

5. Prove the statement made in Example 3. [*Hint*: Problem 4 with $x_0 = 0$.]

6. Let $f(x) = x^k \sin(1/x)$ if $x \neq 0$, and $f(0) = 0$. Show that:
   (a) If $k = 0$, then f is discontinuous at 0.
   (b) If $k = 1$, then f is of class $C^{(0)}$ but not differentiable at 0.
   (c) If $k = 2$, then f is differentiable but not of class $C^{(1)}$.
   (d) What can you say for $k \geq 3$?

7. Let $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.
   (a) If $(x, y) \neq (0, 0)$, find $f_{12}(x, y)$ and $f_{21}(x, y)$ by elementary calculus, and verify that they are equal.
   (b) Using Problem 4 show that $f_1(0, 0) = f_2(0, 0) = 0$ and f is of class $C^{(1)}$.
   (c) Using the definition of partial derivative, show that $f_{12}(0, 0)$ and $f_{21}(0, 0)$ exist but are not equal. Why does this not contradict Theorem 3.3?

8. Given n and q, how many solutions of the equation $i_1 + \cdots + i_n = q$ are there with $i_1, \ldots, i_n$ nonnegative integers? With $i_1, \ldots, i_n$ positive integers? What does this say about the number of different qth-order partial derivatives of a function of class $C^{(q)}$?

3.5 Relative extrema

Let A be some subset of E^n and f a function whose domain contains A. Let us consider the problem of minimizing or maximizing $f(\mathbf{x})$ on A.

Definitions. If $\mathbf{x}_0$ is a point of A such that $f(\mathbf{x}_0) \leq f(\mathbf{x})$ for every $\mathbf{x} \in A$, then f has an *absolute minimum* at $\mathbf{x}_0$. The number

$$f(\mathbf{x}_0) = \min\{f(\mathbf{x}) : \mathbf{x} \in A\}$$

is the *minimum value* of f on A. (Of course, there need not be any such point $\mathbf{x}_0$. However, if A is a compact set, then by Theorem 2.5, any continuous function has an absolute minimum at some point of A.) If $f(\mathbf{x}_0) < f(\mathbf{x})$ for every $\mathbf{x} \in A$ except $\mathbf{x}_0$, then f has a *strict absolute minimum at* $\mathbf{x}_0$.