are the sequence space in Example 5, the space $\mathscr{C}(S)$ in Section 2.10, and the L^p spaces in Section 5.13. We refer to the work by Taylor [22] for an introduction to the theory of Banach spaces and their key role in current mathematical analysis.

Problems

1. Let S be a sphere in E^3, with $\tilde{d}(\mathbf{x}, \mathbf{y})$ as in Example 2. Show that $\tilde{d}$ satisfies properties (i), (ii), and (iii) in the definition of metric space.

2. Let S be a metric space, with metric d.
   (a) Show that, for fixed p_0, the function $d(p_0, \quad)$ is continuous on S. (By definition, the value of $d(p_0, \quad)$ at q is $d(p_0, q)$.)
   (b) Let $\delta > 0$. Show that $\{q : d(p_0, q) \leq \delta\}$ is closed and the δ-neighborhood $\{q : d(p_0, q) < \delta\}$ is open. Give an example in which the first of these two sets is not the closure of the second.

3. Let S be a metric space with metric d. Let
$$\tilde{d}(p, q) = \frac{d(p, q)}{1 + d(p, q)}.$$
   Show that $\tilde{d}$ satisfies properties (i), (ii), and (iii) for a metric space.

4. Let $\mathscr{V}$ be as in Example 5.
   (a) Verify Properties (1) through (3) for the norm $\| \quad \|$ in this example.
   (b) Show that $\mathscr{V}$ is a complete metric space.
   (c) For $l = 1, 2, \ldots$ let $\mathbf{e}_l = (0, \ldots, 0, 1, 0, \ldots)$ where $l - 1$ zeros precede the 1. Let $A = \{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$. Show that A is bounded and closed.
   (d) Show that the set A in part (c) is not compact. [*Hint*: Let $U_l = \{\mathbf{x} : \|\mathbf{x} - \mathbf{e}_l\| < 1\}$. The collection $\{U_1, U_2, \ldots\}$ covers A.]

5. For any compact subsets A, B of E^n, let $d(A, B)$ be the smallest number a with the following property: for every $\mathbf{x} \in B$ there exists $\mathbf{y} \in A$ such that $|\mathbf{x} - \mathbf{y}| \leq a$, and for every $\mathbf{y} \in A$ there exists $\mathbf{x} \in B$ such that $|\mathbf{x} - \mathbf{y}| \leq a$. Show d is a metric on the space whose elements are all compact subsets of E^n.

6. Let S be a compact metric space, with metric d.
   (a) Show that for any $\delta > 0$ there is a finite set $S_\delta \subset S$ such that any $p \in S$ is distant less than δ from some $q \in S_\delta$ [i.e., $d(p, q) < \delta$].
   (b) A set $A \subset S$ is *countable* if either A is a finite set or $A = \{p_1, p_2, \ldots\}$ for some infinite sequence $[p_m]$ in S, $p_m \neq p_l$ for $l \neq m$. Show that there is a countable set A with cl $A = S$.

7. A topological space S_0 is called a *Hausdorff* space if S_0 has the property that for every $p, q \in S_0$ $(p \neq q)$ there exist a neighborhood U of p and a neighborhood V of q such that $U \cap V$ is empty.
   (a) Show that any metric space is a Hausdorff space.
   (b) Show that any compact set $S \subset S_0$ is closed, if S_0 is a Hausdorff space.
   (c) Let f be continuous and univalent from a compact space S onto a Hausdorff space T. Show that f^{-1} is continuous from T onto S. [*Hint*: Show that $(f^{-1})^{-1}(B)$ is closed if B is closed.]

as $m \to \infty$. By Theorem 2.11 f is continuous. Hence $f \in \mathscr{C}(S)$. Thus, any Cauchy sequence $[f_m]$ in $\mathscr{C}(S)$ has a limit f in $\mathscr{C}(S)$, which proves that $\mathscr{C}(S)$ is complete. □

PROBLEMS

1. Show directly from the definition of uniform convergence that the sequence $[f_m]$ in Example 1 does not converge uniformly.
2. Let $f_m(x) = m^2 x$ if $0 \le x \le m^{-1}$, $f_m(x) = m(2 - mx)$ if $m^{-1} \le x \le 2m^{-1}$, and $f_m(x) = 0$ for all other x.
   (a) Find $\| f_m \|$.
   (b) Show that f_m tends to 0 pointwise for each x, but that the convergence is not uniform.
3. Let S be a finite set with n elements. Explain how $\mathscr{C}(S)$ can be identified with the space in Example 4, Section 2.9.
4. Let $f(x) = \sum_{k=1}^{\infty} (\sin kx)/k^2$. Use Theorem 2.11 to show that f is continuous on E^1.
5. Let $S \subset E^n$ be compact, and let $\mathscr{S}$ be a compact subset of $\mathscr{C}(S)$. Show that:
   (a) There exists a number C such that $\|f\| \le C$ for all $f \in \mathscr{S}$.
   (b) Given $\varepsilon > 0$ there exists $\delta > 0$ depending on ε (but not on f), such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ for all $f \in \mathscr{S}$ and all $\mathbf{x}, \mathbf{y} \in S$ satisfying $|\mathbf{x} - \mathbf{y}| < \delta$. [*Hint*: Problem 8, Section 2.5 and Problem 6a, Section 2.9.]

*2.11 Noneuclidean norms on E^n

It is sometimes advantageous to consider norms on E^n other than the standard euclidean norm. The distance between two points $\mathbf{x}$ and $\mathbf{y}$ defined by such a norm need not agree with the euclidean distance. As a result, such geometric notions as length, area, and spherical ball are changed when considered with respect to a noneuclidean norm. However, we see that any noneuclidean norm leads to the same collection of open sets as the euclidean norm. Since the collection of open sets determines all of the topological properties of E^n, these properties are therefore independent of the particular norm chosen.

We recall from Section 2.9 that a norm is a real valued function $\| \ \|$ with domain E^n such that:

(1) $\|\mathbf{x}\| > 0$ for every $\mathbf{x} \neq \mathbf{0}$,
(2) $\|c\mathbf{x}\| = |c| \, \|\mathbf{x}\|$ for every c and $\mathbf{x}$, and
(3) $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ for every $\mathbf{x}$ and $\mathbf{y}$.

The norm defines a metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. The *δ-neighborhood* of $\mathbf{x}_0$ in this metric is $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| < \delta\}$. The *closed n-ball* with center $\mathbf{x}_0$ and radius δ, with respect to the norm $\| \ \|$, is $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| \le \delta\}$. The main result of this section is a characterization of the closed n-ball with center $\mathbf{0}$ and radius 1. By translations and scalar multiplications, this then characterizes all closed n-balls with respect to the norm $\| \ \|$.

This theorem is proved in Section 4.4 as a special case of the composite function theorem for transformations.

In Section 3.3 we define the concept of differentiable function. Since this idea involves linear approximations, we begin in Section 3.2 with linear functions. We see in Section 3.3 that the directional derivatives of a differentiable function are easily computed from the partial derivatives. Disagreeable phenomena of the sort illustrated in Example 2 cannot occur if f is differentiable at $\mathbf{x}_0$.

PROBLEMS

Unless otherwise stated, the domain D of f is E^n for the particular n indicated in the problem.

1. In each case find the partial derivatives of f.
   (a) $f(x, y) = x \log(xy)$, $D = \{(x, y): xy > 0\}$.
   (b) $f(x, y, z) = (x^2 + 2y^2 + z)^3$.
   (c) $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$.
2. Let $f(x, y) = (x - 1)^2 - y^2$. Find the derivative of f at $\mathbf{e}_2$ in any direction $\mathbf{v}$, using the definition of directional derivative.
3. Let $f(x, y) = 2xy(x^2 + y^2)^{-1/2}$, if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Find the derivative of f at $(0, 0)$ in any direction $\mathbf{v}$.
4. Let $f(x, y) = (xy)^{1/3}$. (a) Using the definition of directional derivative, show that $f_1(0, 0) = f_2(0, 0) = 0$, and that $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$ are the only directions in which the derivative at $(0, 0)$ exists. (b) Show that f is continuous at $(0, 0)$.
5. Let $f(x, y, z) = |x + y + z|$. If $\mathbf{x}_0 = (x_0, y_0, z_0)$ is such that $x_0 + y_0 + z_0 = 0$, find those directions $\mathbf{v}$ in which the derivative at $\mathbf{x}_0$ exists.
6. Show that the derivative of f at $\mathbf{x}_0$ in the direction $-\mathbf{v}$ is the negative of the derivative at $\mathbf{x}_0$ in the direction $\mathbf{v}$.

3.2 Linear functions

Let L be a real-valued function whose domain is E^n.

Definition. The function L is *linear* if:

(a) $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in E^n$.
(b) $L(c\mathbf{x}) = cL(\mathbf{x})$ for every $\mathbf{x} \in E^n$ and scalar c.

These two conditions are equivalent to the single condition $L(c\mathbf{x} + d\mathbf{y}) = cL(\mathbf{x}) + dL(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in E^n$ and scalars c, d. By induction, if L is linear,

$$L\left(\sum_{j=1}^{m} c^j \mathbf{x}_j\right) = \sum_{j=1}^{m} c^j L(\mathbf{x}_j) \tag{3.5}$$

give E^n a noneuclidean inner product. This affects the formula for changing covectors into vectors. (See the distinction between the differential and gradient of a function, Section 3.3.)

PROBLEMS

1. Let $n = 3$ and $L(x, y, z) = x + y + 2z$.
   (a) What is the covector $\mathbf{a} = (a_1, a_2, a_3)$ corresponding to L?
   (b) Describe the set $\{(x, y, z) : L(x, y, z) = c\}$ and the intersection of this set with the plane $\{(x, y, z) : y = x\}$.
2. Prove that any linear function L is continuous using Proposition 3.1 and the inequality $|\mathbf{a} \cdot \mathbf{z}| \leq |\mathbf{a}|\,|\mathbf{z}|$
3. Let L be linear on E^n.
   (a) Show that $\{\mathbf{x} : L(\mathbf{x}) = c\}$ is a hyperplane, unless $L(\mathbf{x}) = 0$ for all $\mathbf{x} \in E^n$.
   (b) Give another proof (not the one suggested in Problem 5, Section 1.4) that hyperplanes and closed half-spaces are closed sets, using Problem 2.
4. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for E^n. Define L by the formula
   $$L(c^1\mathbf{x}_1 + \cdots + c^n\mathbf{x}_n) = c^n$$
   for every $c^1, \ldots, c^n$. Show that:
   (a) L is a linear function.
   (b) The set $P = \{\mathbf{x} : L(\mathbf{x}) = 0\}$ is the $(n - 1)$-dimensional vector subspace spanned by $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$.
5. Let $\| \ \|$ be any norm on E^n (Section 2.11). Define on $(E^n)^*$ the *dual norm* as follows. For every covector $\mathbf{a}$,
   $$\|\mathbf{a}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{x}\| = 1\}.$$
   (a) Verify that the dual norm satisfies Properties (1), (2), and (3) (Section 2.9).
   (b) Show that $\|\mathbf{x}\| = \max\{\mathbf{a} \cdot \mathbf{x} : \|\mathbf{a}\| = 1\}$. [*Hint*: Problem 7, Section 2.4.]

3.3 Differentiable functions

The existence of a derivative for a function of one variable is a fact of considerable interest. Geometrically, it says that a tangent line exists. However, the fact that a function of several variables has partial derivatives is not in itself of much interest. For one thing, the existence of derivatives in the directions of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ does not imply that derivatives exist in other directions. Moreover, the function need not have a tangent hyperplane even if there is a derivative in every direction (see Example 2 below).

We shall now define a more natural notion, that of differentiability. Geometrically, differentiability means the existence of a tangent hyperplane. It will be shown that most of the basic properties of differentiable functions of one variable remain true for differentiable functions of several variables.

Let us again consider an interior point $\mathbf{x}_0$ of the domain D of a real valued function f.