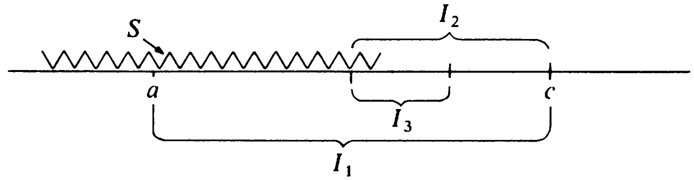

Figure 2.2

$I_1, I_2, \ldots$ as follows: Let a be some point of S, and $I_1 = [a, c]$. Divide I_1 at the midpoint $(a + c)/2$ into two congruent closed intervals. If $(a + c)/2$ is an upper bound for S, let I_2 be the left-hand interval, otherwise let I_2 be the right-hand interval. In general, suppose $m \geq 1$ and I_m has been defined. If the midpoint of I_m is an upper bound for S, let I_{m+1} be the left half of I_m, otherwise the right half. The archimedean property implies that for any $x \geq 0$, the sequence $[m^{-1}x]$ tends to 0 as $m \to \infty$. Since $0 \leq 2^{-m} \leq m^{-1}$, the sequence $[2^{-m}x]$ also tends to 0. Let $x = 2(c - a)$. Now $I_1 \supset I_2 \supset \cdots$ and the length of I_m is $2^{-m}x$. By Theorem 2.3, $I_1 \cap I_2 \cap \cdots$ contains a single point x_0. By the construction, $x_0 = \sup S$ (see Figure 2.2).

Infinite series

Formally, an infinite series is an expression written $\sum_{k=1}^{\infty} \mathbf{x}_k$ or $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$. To be more precise, with any sequence $[\mathbf{x}_k]$ is associated another sequence $[\mathbf{s}_m]$, where $\mathbf{s}_m = \mathbf{x}_1 + \cdots + \mathbf{x}_m$ is called the mth *partial sum*. This pair of sequences defines an *infinite series*. If the sequence of partial sums has a limit $\mathbf{s}$, then the series is *convergent* and $\mathbf{s}$ is its *sum*. This is denoted by $\mathbf{s} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots$. If the sequence of partial sums has no limit, then the series is *divergent*.

If $\mathbf{s} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots$, $\mathbf{t} = \mathbf{y}_1 + \mathbf{y}_2 + \cdots$, then $\mathbf{s} + \mathbf{t} = (\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2) + \cdots$ and $c\mathbf{s} = (c\mathbf{x}_1) + (c\mathbf{x}_2) + \cdots$ for any scalar c. This follows from the definition and Proposition 2.6. Some further elementary properties are given in Problems 7(c) and 8.

Problems

In Problems 1 and 2 you may use the results of Problems 9 and 10.

1. Find the limit if it exists.
(a) $x_m = (2^m - 2^{-m})/(3^m + 3^{-m})$.
(b) $x_m = \sin(m\pi/2)$.
(c) $x_m = \sin m\pi$.
(d) $x_m = ((m + 1)/(m - 1))^m$. [*Hint*: $(1 + 1/m)^m \to e$ as $m \to \infty$.]
(e) $x_m = ((m^2 + 1)/(m^2 - 1))^m$.

2. Find the limit if it exists, using Proposition 2.
(a) $(x_m, y_m) = ((1 + m)/(1 - 2m), 1/(1 + m))$.
(b) $(x_m, y_m) = (2^{-m}, 1 + m)$.
(c) $(x_m, y_m) = (1 - 2^{-m}, (m^2 + 3^m)/m!)$.

3. Show that a sequence $[\mathbf{x}_m]$ has at most one limit $\mathbf{x}_0$. [*Hint*: If $\mathbf{y}_0$ were another limit, let $\varepsilon = |\mathbf{x}_0 - \mathbf{y}_0|/2$.]

4. (a) Let A be a closed set. Show that if $\mathbf{x}_m \in A$ for $m = 1, 2, \ldots$ and $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$, then $\mathbf{x}_0 \in A$.
   (b) If A is not closed, show that there exists a sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in A$ for $m = 1, 2, \ldots$, converging to a limit $\mathbf{x}_0 \notin A$.

5. Show that if $\mathbf{x}_m \in A$ for every $m \geq l$ and $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$, then $\mathbf{x}_0 \in \operatorname{cl} A$.

6. (a) Prove Proposition 2.6.
   (b) Prove Proposition 2.7.

7. (*Comparison tests.*) Show that:
   (a) If $0 \leq x_m \leq y_m$ for every $m \geq l$ and $y_m \to 0$ as $m \to \infty$, then $x_m \to 0$ as $m \to \infty$.
   (b) If $[x_m]$, $[y_m]$ are nondecreasing sequences such that $x_m \leq y_m$ for each $m = 1, 2, \ldots$ and $y_m \to y$ as $m \to \infty$, then $[x_m]$ has a limit $x \leq y$.
   (c) If $0 \leq x_m \leq y_m$ for every $m = 1, 2, \ldots$ and $t = y_1 + y_2 + \cdots$, then the series $x_1 + x_2 + \cdots$ converges with sum $s \leq t$.

8. An infinite series $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$ converges *absolutely* if the series of nonnegative numbers $|\mathbf{x}_1| + |\mathbf{x}_2| + \cdots$ converges. Prove that any absolutely convergent infinite series is convergent. [*Hint*: Show that the sequence $[\mathbf{s}_m]$ of partial sums is Cauchy.]

9. Show that if $a > 0$, then
   (a) $\lim_{m\to\infty} a^{1/m} = 1$. (b) $\lim_{m\to\infty} a^m/m! = 0$.
   (c) $\lim_{m\to\infty}(x_m)^{1/m} = 1$ provided $\lim_{m\to\infty} x_m = a$.
   [*Hints*: For part (a) reduce to the case $0 < a < 1$. By Example 1, if $b < 1$ then $a \leq b^m$ for only finitely many m. For part (b), compare with the sequence $[c/m]$ for suitable c and suitable l in Problem 7(a).]

10. Let $x_0 = \lim_{m\to\infty} x_m$, $y_0 = \lim_{m\to\infty} y_m$, and assume that $y_m \neq 0$ for $m = 0, 1, 2, \ldots$. Show that $x_0/y_0 = \lim_{m\to\infty} x_m/y_m$. [*Hint*: By (c) of Proposition 2.6 it suffices to show that $y_0^{-1} = \lim_{m\to\infty} y_m^{-1}$.]

11. Show that $x_0 = y_0$ in Example 3.

2.4 Bolzano–Weierstrass theorem

Suppose that an infinite number of points lie in a box. It is intuitively reasonable that they cannot remain scattered but must accumulate at some points of the box. The purpose of the present section is to put this idea on a precise basis. We begin with definitions of the concepts of isolated point and accumulation point of a set $A \subset E^n$.

Definition. A point $\mathbf{x}_0$ is an *isolated point* of A if there exists a neighborhood U of $\mathbf{x}_0$ such that $A \cap U = \{\mathbf{x}_0\}$.

Definition. A point $\mathbf{x}_0$ is an *accumulation point* of A if every neighborhood of $\mathbf{x}_0$ contains an infinite number of points of A.

Let us consider the case when A is an infinite set. Then $A \subset A_1$ since each $A_m \subset A_1$. Since A_1 is bounded, A is bounded. Let $\mathbf{x}_0$ be an accumulation point of A. As in the proof of Corollary 1, we have $\mathbf{x}_0 \in A_1$ since A_1 is a closed set and $A \subset A_1$. For each $m = 1, 2, \ldots, \mathbf{x}_0$ is also an accumulation point of the set $\{\mathbf{x}_m, \mathbf{x}_{m+1}, \ldots\}$. Since this set is contained in A_m and A_m is closed, we get in the same way $\mathbf{x}_0 \in A_m$. Since this is true for each m, $\mathbf{x}_0 \in \bigcap_{m=1}^{\infty} A_m$. □

Corollary 3 shows the existence of a point in any closed set A nearest a given point $\mathbf{x}_0 \notin A$.

Corollary 3. *Let A be a closed, nonempty subset of E^n and $\mathbf{x}_0 \notin A$. Then there exists $\mathbf{x}_1 \in A$ such that $|\mathbf{x} - \mathbf{x}_0| \geq |\mathbf{x}_1 - \mathbf{x}_0|$ for all $\mathbf{x} \in A$.*

Proof. Let $S_r = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ denote the closed spherical n-ball with center $\mathbf{x}_0$ and radius r. Let

$$d = \inf\{|\mathbf{x} - \mathbf{x}_0| : \mathbf{x} \in A\}$$

$$d_m = d + \frac{1}{m}, \quad m = 1, 2, \ldots$$

$$A_m = A \cap S_{d_m}.$$

The sets $A_1, A_2, \ldots$ satisfy the hypotheses of Corollary 2. Let $\mathbf{x}_1 \in \bigcap_{m=1}^{\infty} A_m$. Then $\mathbf{x}_1 \in A$ since each $A_m \subset A$. By definition of d, $|\mathbf{x}_1 - \mathbf{x}_0| \geq d$. Since $\mathbf{x}_1 \in A_m$, and $A_m \subset S_{d_m}$, $|\mathbf{x}_1 - \mathbf{x}_0| \leq d + 1/m$ for each $m = 1, 2, \ldots$. Thus $|\mathbf{x}_1 - \mathbf{x}_0| = d$. □

Note. The point $\mathbf{x}_1$ in Corollary 3 need not be unique. However, $\mathbf{x}_1$ is unique if A is also convex (Problem 3).

Problems

1. Find all accumulation points of A:
   (a) $A = \{(-1)^m m(1 + m)^{-1} : m = 1, 2, \ldots\}$.
   (b) $A = \left\{\left(\cos\frac{2m\pi}{5}, \sin\frac{2m\pi}{5}\right) : m = 1, 2, \ldots\right\}$.
   (c) $A = \left\{\left(\left(1 - \frac{1}{m}\right)\cos\frac{2m\pi}{5}, \left(1 - \frac{1}{m}\right)\sin\frac{2m\pi}{5}\right) : m = 1, 2, \ldots\right\}$.
   (d) $A = \{(x, y) : (x^2 + y^2)(y^2 - x^2 + 1) \leq 0\}$.
   (e) $A = \{\cos m : m = 1, 2, \ldots\}$.
2. Prove Proposition 2.9.
3. Let A be a closed, convex, nonempty set, and $\mathbf{x}_0 \notin A$. Show that there is exactly one point $\mathbf{x}_1 \in A$ nearest $\mathbf{x}_0$.

4. A set A is called *dense in* B if every point of B is an accumulation point of A.
(a) Suppose that A has no isolated points. Show that A is dense in B if and only if $B \subset \text{cl } A$.
(b) Suppose that A is dense in B and B is dense in C. Show that A is dense in C.

5. Let $C = [0, 1] - (A_1 \cup A_2 \cup \cdots)$, where $A_1 = (\frac{1}{3}, \frac{2}{3})$, $A_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, $A_3 = (\frac{1}{27}, \frac{2}{27}) \cup \cdots \cup (\frac{25}{27}, \frac{26}{27})$, and A_j is the union of 2^{j-1} open intervals of length 3^{-j} chosen similarly (see Figure 2.4). [*Note*: C is called the *Cantor set*.]
(a) Show that C is a closed set.
(b) Show that C is dense in no open set.

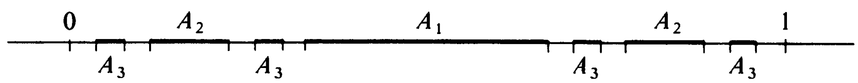

Figure 2.4

6. (*Subsequences.*) Let $[\mathbf{x}_m]$ be a sequence, and $\mathbf{y}_l = \mathbf{x}_{m_l}$ for $l = 1, 2, \ldots$, where $m_1 < m_2 < \ldots$. Then $[\mathbf{y}_l]$ is called a *subsequence* of $[\mathbf{x}_m]$.
(a) Show that any bounded sequence in E^n has a convergent subsequence.
(b) A set S is called *sequentially compact* if: any bounded sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$, has a subsequence $[\mathbf{y}_l]$ such that $\mathbf{y}_l \to \mathbf{y}_0$ as $l \to \infty$, $\mathbf{y}_0 \in S$. Show that a nonempty set $S \subset E^n$ is sequentially compact if and only if S is closed and bounded.

7. Let $\mathbf{y}$ be any frontier point of a closed convex set K. Show that K has a supporting hyperplane P that contains $\mathbf{y}$. [*Hint*: Let $\{\mathbf{y}_m\}$ be a sequence of points exterior to K such that $\mathbf{y}_m$ tends to $\mathbf{y}$ as $m \to \infty$. Let $\mathbf{x}_m$ be a point of K nearest to $\mathbf{y}_m$ and

$$\mathbf{u}_m = \frac{\mathbf{y}_m - \mathbf{x}_m}{|\mathbf{y}_m - \mathbf{x}_m|}.$$

Then $|\mathbf{u}_m| = 1$ and $\mathbf{x}_m$ tends to $\mathbf{y}$ as $m \to \infty$. By the proof of Theorem 1.1 there is a supporting hyperplane of the form $\{\mathbf{x} : \mathbf{u}_m \cdot (\mathbf{x} - \mathbf{x}_m) = 0\}$. Let $\mathbf{u}$ be an accumulation point of the bounded set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ and $P = \{\mathbf{x} : \mathbf{u} \cdot (\mathbf{x} - \mathbf{y}) = 0\}$.]

2.5 Relative neighborhoods, continuous transformations

For the definition in Section 2.2 of a transformation continuous at a point $\mathbf{x}_0$, it is assumed that $\mathbf{x}_0$ is interior to the domain D. However, we often wish to discuss continuity at points which are not interior to the domain. Moreover, even if the domain D is an open subset of E^n, we may be interested only in the restriction of the transformation to some set $S \subset D$.

We easily circumvent this apparent difficulty by introducing the idea of relative neighborhood.

Definition. Let S be a nonempty subset of E^n. A *relative neighborhood* of a point $\mathbf{x} \in S$ is any set U such that $U = S \cap W$, where W is a neighborhood of $\mathbf{x}$ in E^n (Figure 2.5).

Next suppose that $\mathbf{f}(S)$ is not closed. Then there exists $\mathbf{y}_0 \in \text{cl}[\mathbf{f}(S)] - \mathbf{f}(S)$. For $m = 1, 2, \ldots$ there exists $\mathbf{y}_m \in \mathbf{f}(S)$ such that $\mathbf{y}_m \to \mathbf{y}_0$ as $m \to \infty$. Choose $\mathbf{z}_m \in S$ with $\mathbf{y}_m = \mathbf{f}(\mathbf{z}_m)$. As before, the set $B = \{\mathbf{z}_1, \mathbf{z}_2, \ldots\}$ has an accumulation point $\mathbf{z}_0 \in S$. Since $\mathbf{y}_0 \notin \mathbf{f}(S)$, $\mathbf{y}_0 \neq \mathbf{f}(\mathbf{z}_0)$. Let V be a neighborhood of $\mathbf{f}(\mathbf{z}_0)$ of radius $\frac{1}{2}|\mathbf{f}(\mathbf{z}_0) - \mathbf{y}_0|$. Since $\mathbf{f}$ is continuous at $\mathbf{z}_0$, there exists a relative neighborhood U of $\mathbf{z}_0$ with $\mathbf{f}(U) \subset V$. In particular, $\mathbf{y}_m = \mathbf{f}(\mathbf{z}_m)$ is in V for infinitely many m for which $\mathbf{z}_m \in U$. This contradicts the fact that $\mathbf{y}_m \to \mathbf{y}_0$ as $m \to \infty$. Thus $\mathbf{f}(S)$ is closed. □

If we specialize Theorem 2.4 to real valued functions, we get a theorem about the existence of maxima and minima.

Theorem 2.5. *Let f be a real valued function continuous on a closed, bounded set S. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that*

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2)$$

for all $\mathbf{x} \in S$.

PROOF. We recall that if $T \subset E^1$ is bounded and closed, then $y_1 = \inf T$ and $y_2 = \sup T$ are points of T (Example 4, Section 1.4). Let $T = f(S)$. By Theorem 2.4, T is closed and bounded. Take $\mathbf{x}_i$ such that $y_i = f(\mathbf{x}_i)$, $i = 1, 2$. □

The function f is said to have a *minimum* on S at $\mathbf{x}_1$ and a *maximum* on S at $\mathbf{x}_2$.

PROBLEMS

1. In each case show that f has a minimum on S, but no maximum on S. Which assumption in Theorem 2.5 is violated?
   (a) $S = (0, 1]$, $f(x) = x^{-1}$.
   (b) $S = E^n$, $f(\mathbf{x}) = \dfrac{|\mathbf{x}|}{1 + |\mathbf{x}|}$.

2. Given $\mathbf{x}_0$, let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|$. Show that f has a minimum on any closed, nonempty set $A \subset E^n$. (This gives another proof of Corollary 3, Section 2.4.)

3. Let $S = [a, b]$.
   (a) Show that the relative neighborhoods of a are the half open intervals $[a, c)$ with $a < c < b$, and S.
   (b) Let $a < x_0 < \frac{1}{2}(a + b)$. Show that the relative neighborhoods of x_0 are as follows: $(x_0 - \delta, x_0 + \delta)$ if $0 < \delta < x_0 - a$; $[a, x_0 + \delta)$ if $x_0 - a \leq \delta < b - x_0$; S.
   (c) Describe the relative neighborhoods in the remaining cases $\frac{1}{2}(a + b) \leq x_0 \leq b$.

4. Define the projection π from E^n onto E^s as in Problem 7, Section 2.1.
   (a) Show that $\pi(A)$ is closed and bounded if A is closed and bounded.
   (b) Give an example of a closed set A such that $\pi(A)$ is not closed.

5. (a) Let $\mathbf{f}$ be continuous on S, and let $S_1 \subset S$. Show that the restriction $\mathbf{f}|S_1$ is continuous on S_1.
(b) Let $f(x) = 1 - x$ if $x \geq 0$ and $f(x) = 0$ if $x < 0$. Let $S_1 = [0, \infty)$, $S = E^1$. Show that $f|S_1$ is continuous, but f is not continuous at each point of S_1.

6. Let $\mathbf{f}$ be a transformation with domain S. Show that $\mathbf{f}$ is continuous at $\mathbf{x}_0$ if and only if $\mathbf{f}(\mathbf{x}_0) = \lim_{m \to \infty} \mathbf{f}(\mathbf{x}_m)$ for every sequence $[\mathbf{x}_m]$ such that $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots$ and $\mathbf{x}_m \to \mathbf{x}_0$ as $m \to \infty$.

7. Let f be continuous on E^n. Suppose, moreover, that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and that $f(c\mathbf{x}) = cf(\mathbf{x})$ for any $\mathbf{x}$ and $c > 0$. Show that there exist $a > 0$ and $b > 0$ such that $a|\mathbf{x}| \leq f(\mathbf{x}) \leq b|\mathbf{x}|$. [*Hint*: First consider $\{\mathbf{x} : |\mathbf{x}| = 1\}$.]

8. (*Uniform continuity.*) A transformation $\mathbf{f}$ is *uniformly continuous* on $S \subset E^n$ if given $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$ for every $\mathbf{x}, \mathbf{y} \in S$ with $|\mathbf{x} - \mathbf{y}| < \delta$. Show that if S is closed and bounded then every $\mathbf{f}$ continuous on S is uniformly continuous on S. [*Hint*: If not, then there exists $\varepsilon > 0$ and for $m = 1, 2, \ldots, \mathbf{x}_m, \mathbf{y}_m \in S$ such that $|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{y}_m)| \geq \varepsilon$ and $|\mathbf{x}_m - \mathbf{y}_m| \leq 1/m$. Let $\mathbf{x}_0$ be an accumulation point of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. Show that the continuity of $\mathbf{f}$ at $\mathbf{x}_0$ is contradicted.]

2.6 Topological spaces

In order to proceed further with the study of subsets of E^n and continuous transformations, it is convenient to introduce a very general concept—that of topological space. In this section S denotes a set, not necessarily a subset of E^n, and p denotes a point of S.

The notion of topological space occurs in practically all branches of mathematics. There are several equivalent definitions; of these, we give the one in terms of neighborhoods.

Definition. Let S be a nonempty set. For every $p \in S$ let $\mathscr{U}_p$ be a collection of subsets of S called *neighborhoods* of p such that:

(1) Every point p has at least one neighborhood.
(2) Every neighborhood of p contains p.
(3) If U_1 and U_2 are neighborhoods of p, then there is a neighborhood U_3 of p such that $U_3 \subset U_1 \cap U_2$.
(4) If U is a neighborhood of p and $q \in U$, then there is a neighborhood V of q such that $V \subset U$.

Then S is a *topological space*.

More precisely, the topological space is S together with the collections $\mathscr{U}_p$ of neighborhoods. However, it is common practice to omit explicit reference to the collections of neighborhoods when no ambiguity can arise.

For our purposes, the following two examples of topological spaces are of primary importance.