HW4 MTH 428/528 Spring 2025

Sunday, February 9, 2025 9:20 AM

#2 Find limit if it exists

(a) $\lim_{m\to\infty}\left(\frac{1+m}{1-2m}, \frac{1}{1+m}\right)$

$= \left(\lim_{m\to\infty}\frac{1+m}{1-2m}, \lim_{m\to\infty}\frac{1}{1+m}\right)$

$= \left(-\frac{1}{2}, 0\right)$

(b) $\lim_{m\to\infty}\left(2^{-m}, 1+m\right)$

Does not exist since $\lim_{m\to\infty} 1+m$ DNE

(c) $\lim_{m\to\infty}\left(1-2^{-m}, \frac{m^2+3m^3}{m!}\right)$

$= \left(\lim_{m\to\infty} 1-\frac{1}{2^m}, \lim_{m\to\infty}\frac{m^2+3m^3}{m!}\right)$ facterial faster than any polynomial!

$= (1-0, 0)$

$= (1, 0)$

#3 Show a sequence $\{x_m\}$ has at most one limit.

Proof: Suppose both a and b are limits of $\{x_m\}$ with $a \neq b$. Let $\epsilon = \frac{\|a-b\|}{2}$. By def of limit, $\exists N_a, N_b > 0$ s.t. $\forall n > N_a$ and $\forall m > N_b$,

$\|x_n - a\| < \epsilon = \frac{\|a-b\|}{2}$ and

$\|x_m - b\| < \epsilon = \frac{\|a-b\|}{2}$

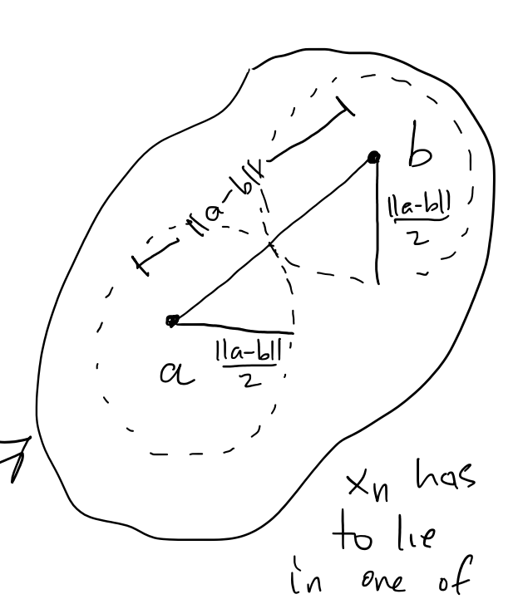
Let $N = \max\{N_a, N_b\}$ and let $l > N$.

Then we simultaneously have

$\|x_l - a\| < \frac{\|a-b\|}{2}$ and $\|x_l - b\| < \frac{\|a-b\|}{2}$

But this is impossible!

Therefore only one limit can exist.



x_n has to lie in one of the circles — it can't lie in both!

$\forall\epsilon>0\,\exists N\,\forall n>N\ \|x_0 - x_m\| < \epsilon$ $\quad$ $\forall\epsilon>0\,\exists N\,\forall n>N\ |x_0^i - x_m^i| < \epsilon$

#6(b) Prove: $x_0 = \lim_{m\to\infty} x_m$ iff $x_0^i = \lim_{m\to\infty} x_m^i$ for each $i = 1, 2, \ldots, n$

Proof: $(\longrightarrow)$ We assume here that

$\forall\epsilon>0\,\exists N\,\forall n>N\ \|x_n - x_0\| = \sqrt{\sum_{j=1}^{n}(x_n^j - x_0^j)^2} < \epsilon$

But,

$\epsilon > \|x_n - x_0\| = \sqrt{\sum_{j=1}^{n}(x_n^j - x_0^j)^2} \geq \sqrt{(x_n^i - x_0^i)^2} = |x_n^i - x_0^i|$

Thus

$\forall\epsilon>0\,\exists N\,\forall n>N\ |x_n^i - x_0^i| < \epsilon$, completing the proof.

$(\longleftarrow)$ Assume for each $i = 1, \ldots, n$ that $\lim_{m\to\infty} x_m^i = x_0^i$

So, $\forall\epsilon>0\,\exists N_i\,\forall n>N_i\ |x_n^i - x_0^i| < \frac{\epsilon}{\sqrt{n}}$.

Let $N = \max\{N_1, N_2, \ldots, N_n\}$ and let $n > N$.

Now compute

$\|x_n - x_0\| = \sqrt{\sum_{i=1}^{n}(x_n^i - x_0^i)^2} < \sqrt{\sum_{i=1}^{n}\left(\frac{\epsilon}{\sqrt{n}}\right)^2}$

$= \sqrt{\frac{\epsilon^2}{n}\boxed{\sum_{i=1}^{n} 1}}$ $= n$

$= \sqrt{\epsilon^2} = \epsilon$,

completing the proof. ∎

#8 Prove any absolutely convergent series is convergent.

Proof: Suppose $\sum_{k=1}^{\infty} x_k$ is absolutely convergent, i.e. that $\sum_{k=1}^{\infty}|x_k|$ converges, which means the partial sums $s_n = \sum_{k=1}^{n}|x_k|$ converge to some limit S, i.e.

$\forall\epsilon>0\,\exists N\,\forall n>N\ |S - s_n| < \epsilon$.

By Thm 2.2 (Cauchy criterion) we know that the sequence $\{s_n\}$ is a Cauchy sequence, i.e.

$\forall\epsilon>0\,\exists N\,\forall n,m>N\ |s_n - s_m| < \epsilon$.

Let $\epsilon > 0$ and choose N so that $\forall n,m > 0$, $|s_n - s_m| < \epsilon$.

Let $t_n = \sum_{k=1}^{n} x_k$ (the partial sums for $\sum_{k=1}^{\infty} x_k$).

Let $m, n > N$ (WLOG assume $n > m$)

$|t_n - t_m| = \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right|$

$= |(x_1 + x_2 + \cdots + x_n) - (x_1 + x_2 + \cdots + x_m)|$

$\overset{n>m}{=} |x_{m+1} + x_{m+2} + \cdots + x_n|$

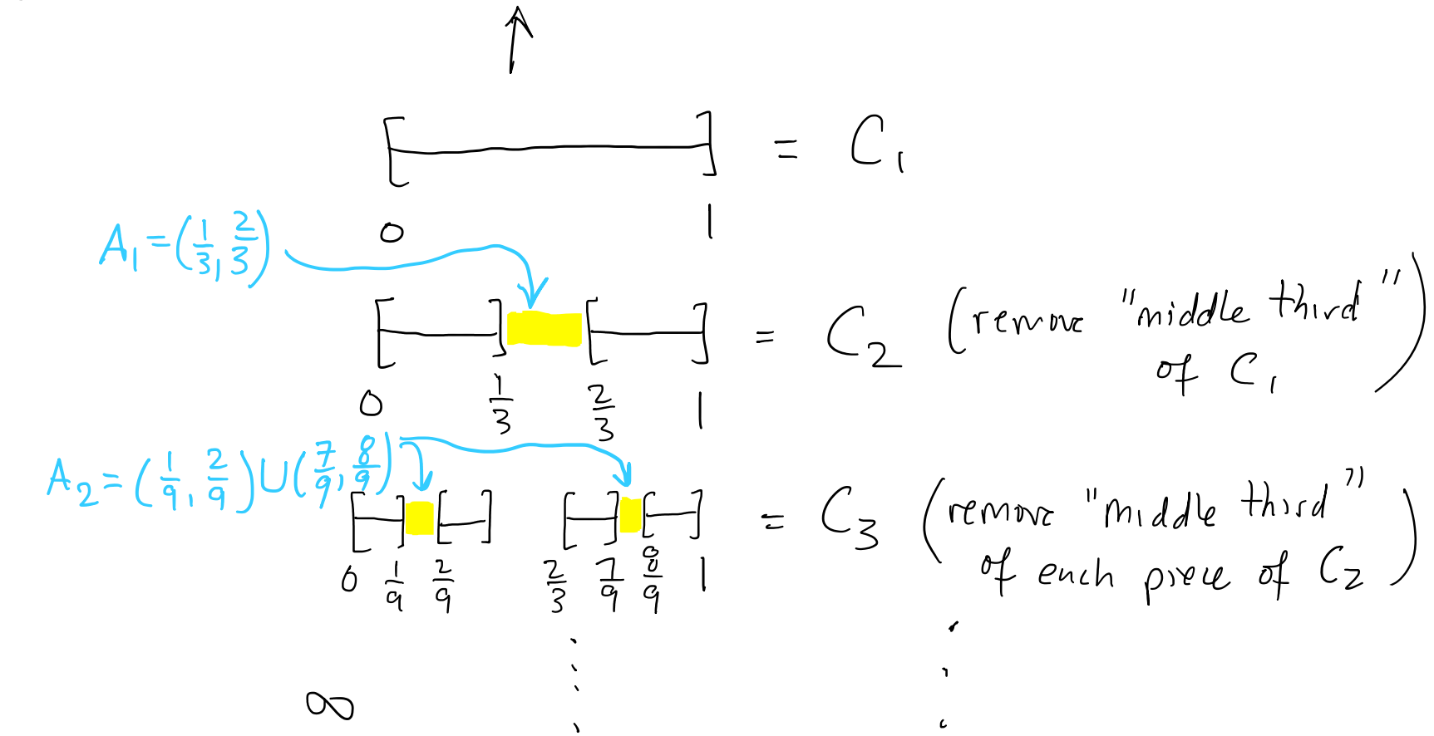
$\overset{\Delta\text{-inequality}}{\leq} |x_{m+1}| + |x_{m+2}| + \cdots + |x_n|$

$= \left|\sum_{k=1}^{n}|x_k| - \sum_{k=1}^{m}|x_k|\right|$

$= |s_n - s_m| < \epsilon$,

This shows $\{t_n\}$ is a Cauchy sequence and so by Cauchy criterion, it converges, completing the proof. ∎

§2.4 #5(a) Let $\mathcal{C}$ be the Cantor set



Here $\mathcal{C} = \bigcap_{k=1}^{\infty} C_k$

Show $\mathcal{C}$ is a closed set.

Proof: Recall that an arbitrary union of open sets is open: $\bigcup \mathcal{U}_\alpha$

Also recall the complement of an open set is a closed set. So,

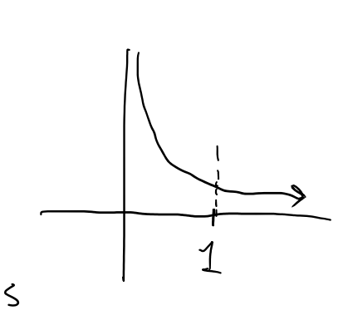
$\left(\bigcup \mathcal{U}_\alpha\right)^c = \bigcap \mathcal{U}_\alpha^c$ is an intersection of closed sets which is closed.

Since each C_k in the Cantor set is a finite union of closed sets, each C_k is closed. Thus $\mathcal{C}$ being an intersection of closed sets, we get $\mathcal{C}$ is closed. ∎

§2.5 #1 Show f has a minimum on S but no maximum. Which assumption of $\boxed{\text{Thm 2.5}}$ is violated?

f real-valued fnct on closed bdd set S has a max value and a min value

(a) $S = (0, 1]$, $f(x) = \frac{1}{x}$



Soln: Here the minimum occurs at $x = 1$.

Thm 2.5 does not apply because the domain is not closed!

(b) $S = E^n$, $f(x) = \frac{\|x\|}{1+\|x\|}$

Soln: If $x = 0$, we see that $f(0) = 0$ must be the minimum because $f(x) \geq 0$ for all x.

Thm 2.5 does not apply b/c the domain of f is not bounded!

Graduate student problems

§2.3 #4(a) Let A be closed. Show if $x_m \in A$ for $m = 1, 2, \ldots$ and $x_0 = \lim_{m\to\infty} x_m$, then $x_0 \in A$.

Soln: Let $\epsilon > 0$ and consider the open ball centered at x_0 of radius ϵ:

$B_\epsilon = \{x : \|x - x_0\| < \epsilon\}$

But since $x_0 = \lim_{m\to\infty} x_m$, it means that

$\exists N\,\forall n>N\ \|x_n - x_0\| < \epsilon$.

This means $x_{n+1}, x_{n+2}, \ldots \in B_\epsilon$. So $B_\epsilon \cap A$ is nonempty for every $\epsilon > 0$. So, x_0 is a limit point of A.

So $x_0 \in A$ since A is closed. ∎

#5 Show if $x_m \in A$ for every $m \geq l$ and $x_0 = \lim_{m\to\infty} x_m$, then $x_0 \in cl(A)$.

Proof: We know $\forall\epsilon>0\,\exists N\,\forall n>N\ \|x_0 - x_n\| < \epsilon$.

So for any $\epsilon > 0$, the ball $B_\epsilon = \{x : \|x - x_0\| < \epsilon\}$ contains ∞-many of the points of the sequence.

Thus, x_0 is a limit point of A, so $x_0 \in cl(A) = A \cup \underbrace{fr(A)}$. ∎

limit points (or "frontier points") of A

§2.4 #1 Find all accumulation points of A.

(a) $A = \left\{\frac{(-1)^m m}{1+m} ; m = 1, 2, 3, \ldots\right\}$

Soln: Since $\frac{m}{1+m} \xrightarrow{m\to\infty} 1$, we see that the sequence is oscillating between a sequence converging to 1 and one converging to -1. So the accumulation points are 1 and -1.

(b) $A = \left\{\overbrace{\left(\cos\left(\frac{2m\pi}{5}\right), \sin\left(\frac{2m\pi}{5}\right)\right)}^{\text{call this } w_m}, m = 1, 2, 3, \ldots\right\}$

Soln: Since cosine and sine are 2π-periodic, i.e.

$\cos(\theta + 2\pi n) = \cos(\theta)$ and $\sin(\theta + 2\pi n) = \sin(\theta)$

and

$\frac{2m\pi}{5} + 2n\pi = \frac{2m\pi}{5} + \frac{10n\pi}{5} = \frac{2\pi(m+5n)}{5}$

we see that there are distinct values of w_m for $m = 0, 1, 2, 3, 4$ and then it repeats itself.

So, the accumulation points are

m	w_m
0	$((\cos(0), \sin(0)) = (1, 0)$
1	$\left(\cos\left(\frac{2\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right)\right)$
2	$\left(\cos\left(\frac{4\pi}{5}\right), \sin\left(\frac{4\pi}{5}\right)\right)$
3	$\left(\cos\left(\frac{6\pi}{5}\right), \sin\left(\frac{6\pi}{5}\right)\right)$
4	$\left(\cos\left(\frac{8\pi}{5}\right), \sin\left(\frac{8\pi}{5}\right)\right)$

(notice at $m = 5$ we get $w_5 = \left(\cos\left(\frac{10m\pi}{5}\right), \sin\left(\frac{10m\pi}{5}\right)\right)$
$= (\cos(2\pi), \sin(2\pi))$
$= (\cos(0), \sin(0))$)

#4 A is called dense in B if every pt of B is an accumulation point of A

(a) Suppose A contains no isolated points. Show A is dense in B iff $B \subset cl(A)$.

Proof: $(\longrightarrow)$ Suppose A is dense in B. So every point of B is an accumulation point of A, i.e.

for all $x \in B$, every open set containing x also contains infinitely many points of A.

If $x \in A$, then $x \in cl(A) = A \cup fr(A)$. If $x \notin A$, then for any nhbd $\mathcal{U}$ of x, $x \in \mathcal{U}$ and some point of A also lies in $\mathcal{U}$. Thus such $x \in fr(A)$, hence $x \in cl(A) = A \cup fr(A)$. Therefore, $B \subset fr(A)$.

$(\longleftarrow)$ Suppose $B \subset cl(A)$. Let $x \in B$ — we must argue that x is an accumulation point of A.

If $x \in A$, then since A contains no isolated points, x is an accumulation point of A.

If $x \notin A$, then since $x \in cl(A) = A \cup fr(A)$, we conclude that $x \in fr(A)$. So for each $\epsilon_i = \frac{1}{2i}$, $i = 1, 2, 3, \ldots$ the open ball $B_i = \{z : \|x - z\| < \epsilon\}$ has nonempty intersection with A, say $y_i \in B_i \cap A$.

Then for any open $\mathcal{U}$ of x, since $\mathcal{U}$ is open, we get that $\exists M$ s.t. $\forall m > M$, $B_m \subset \mathcal{U}$.

Thus $\{y_m, y_{m+1}, y_{m+2}, \ldots\}$ are ∞-many points of A in $\mathcal{U}$.

Thus x is an accumulation point of A.

So we showed A is dense in B. ∎

(b) Suppose A is dense in B and B is dense in C. Show that A is dense in C.

Proof: Since A is dense in B, all points of B are accumulation points of A. Since B is dense in C, all points of C are accumulation points of B.

We need to show all points of C are accumulation points of A.

Let $x \in C$ and $\mathcal{U}$ be a nhbd of x. We know that $\mathcal{U}$ contains infinitely many points of B, say $\{b_1, b_2, \ldots\}$.

Since $\mathcal{U}$ is open $\forall i = 1, 2, 3, \ldots\ \exists \epsilon_i$ s.t. the ball centered at b_i of radius ϵ_i, $\mathcal{W}_i = \{w : \|b_i - w\| < \epsilon_i\} \subset \mathcal{U}$.

Since each b_i is an accumulation point of A, each $\mathcal{V}_i$ contains ∞-many points of A, and by construction all such points are also in $\mathcal{U}$. So pick one such $a_i \in \mathcal{V}_i$.

Then $\{a_1, a_2, a_3, \ldots\}$ is an infinite set of points of A lying in $\mathcal{U}$.

Thus, A is dense in C. ∎

§2.5 #5

(a) Let f be continuous on S and let $S_1 \subset S$. Show the restriction $f|_{S_1}$ is continuous on S_1.

Proof: Let $x \in S$ and let $\mathcal{V}$ be a nhbd of $f(x)$.

Since f is continuous on S, $\exists$ nhbd $\mathcal{U}$ of x s.t. $f(\mathcal{U}) \subset \mathcal{V}$.

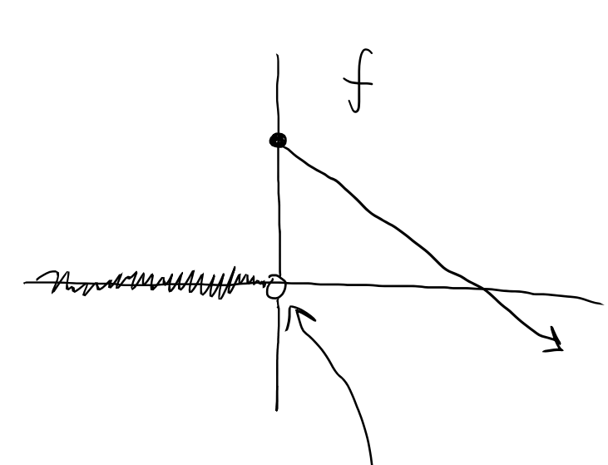
Further assume $x \in S_1$.

The set $\mathcal{U}_R = \mathcal{U} \cap S_1$ is a relative nhbd of x and $f(\mathcal{U}_R) \subset \mathcal{V}$, so f is continuous at x.

Since x was arbitrary, we get that $f|_{S_1}$ is continuous. ∎

(b) Let $f(x) = \begin{cases} 1-x, & x \in [0, \infty) \\ 0, & x \in (-\infty, 0) \end{cases}$

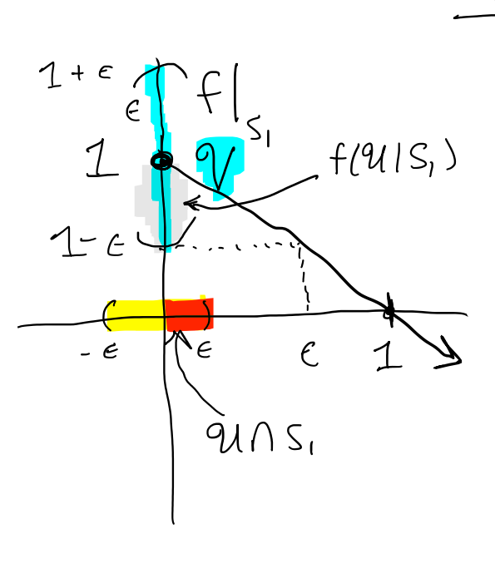
Let $S_1 = [0, \infty)$, $S = \mathbb{R}$



Show $f|_{S_1}$ is continuous but f is not continuous at every point of S_1.

$f(x) = 1 - x = 1 - \epsilon$
$-x = -\epsilon$
$x = \epsilon$

Proof: f is not continuous at $x = 0$ (clearly!)



Let $x \in S_1 \setminus \{0\}$ and since polynomials are continuous, we get f ctn at x.

Now let $x = 0$. Let $\mathcal{V}$ be an open neighborhood of $f(0) = 1$, say $\mathcal{V} = \{w : \|w - 1\| < \epsilon\}$.

If $\epsilon \geq 1$, choose $\mathcal{U} = (-1, 1)$, then $f(\mathcal{U} \cap S_1) \subset \mathcal{V}$.

If $\epsilon < 1$, choose $\mathcal{U} = (-\epsilon, \epsilon)$, then $\mathcal{U} \cap S_1 = [0, \epsilon)$ and we see $f(\mathcal{U}|S_1) \subset \mathcal{V}$.

Therefore $f|_{S_1}$ is continuous even though f was not continuous at 0. ∎