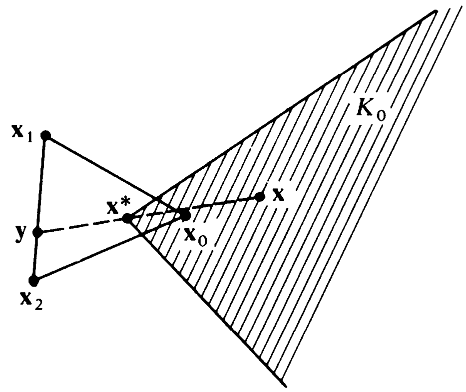

Figure 1.13

$\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the vertices of an n-simplex T, and all the barycentric coordinates of $\mathbf{x}^*$ are positive. Let T_0 be the face of T opposite $\mathbf{x}_0$, and

$$K_0 = \{\mathbf{x} : \mathbf{x}^* = t\mathbf{x} + (1 - t)\mathbf{y}, \text{ where } \mathbf{y} \in T_0 \text{ and } t \in [0, 1]\}.$$

K_0 is a convex polytope, and its boundary fr K_0 consists of portions of the hyperplanes which contain $\mathbf{x}^*$ and the $(n - 2)$-dimensional faces of T_0 (we leave the verification of this to the reader). If fr K_0 intersects S, then $\mathbf{x}^*$ is a convex combination of fewer than $n + 1$ points of S, contrary to hypothesis (Figure 1.13). Hence $S \cap$ fr K_0 is empty. The interior int K_0 and the complement $K_0^c = E^n - K_0$ are open sets, their union contains S, and their intersection is empty. But $\mathbf{x}_0 \in$ int K_0 and $\mathbf{x}_i \in K_0^c$ for $i = 1, \ldots, n$. This implies that S is disconnected, which is a contradiction. □

By slightly refining the proof, an even stronger result is obtained. Suppose that $S = S_1 \cup \cdots \cup S_k$, where $k \leq n$ and $S_1, \ldots, S_k$ are connected sets. For each $i = 1, \ldots, n$ consider the corresponding convex polytope K_i. Then int $K_i \cap$ int K_j is empty whenever $i \neq j$ and $S \cap$ fr K_i is empty for every i. Moreover, $\mathbf{x}_i \in$ int K_i. Since $k \leq n$, some pair of the points $\mathbf{x}_i, \mathbf{x}_j$ must belong to the same set S_p. Then S_p is not connected, a contradiction. Hence, if S is the union of n or fewer connected sets, every $\mathbf{x}$ which is a convex combination of points of S is a convex combination of n or fewer points of S.

Problems

1. Show that each of the following subsets of E^2 is closed and convex by writing it as the intersection of closed half-planes:
   (a) The regular hexagon with center (0, 0) and $\mathbf{e}_1$ as one vertex.
   (b) $\{(x, y) : y \geq |x|, -1 \leq x \leq 1\}$.
   (c) $\{(x, y) : y \leq \log x, x > 0\}$.
   (d) $\{(x, y) : 0 \leq y \leq \sin x, 0 \leq x \leq \pi\}$.

2. Write the standard n-simplex as the intersection of $n + 1$ closed half-spaces. Illustrate for $n = 2$ and $n = 3$.

3. Write $\frac{1}{4}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2$ as a convex combination of $\mathbf{e}_1, \frac{4}{3}\mathbf{e}_2 - \mathbf{e}_1$. Also write it as a convex combination of $\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$. Illustrate.

4. Show that if $\mathbf{x}$ can be represented in two ways as a convex combination of $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r$, then $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_r - \mathbf{x}_0$ form a linearly dependent set. [*Hint*: If $\mathbf{x} = t^0\mathbf{x}_0 + \cdots + t^r\mathbf{x}_r$ and $t^0 + \cdots + t^r = 1$, then $\mathbf{x} - \mathbf{x}_0 = t^1(\mathbf{x}_1 - \mathbf{x}_0) + \cdots + t^r(\mathbf{x}_r - \mathbf{x}_0)$.]

5. Prove that a supporting hyperplane for a closed convex set K can contain no interior point of K.

6. Let K be any convex set. Prove that its interior and its closure are also convex sets.

7. The *barycenter* of an r-simplex is the point at which the barycentric coordinates are equal, $t^0 = t^1 = \cdots = t^r$.
(a) Show that the barycenter of a triangle is at the intersection of the medians.
(b) State and prove a corresponding result for $r \geq 3$.

8. Let $\mathbf{x}$ be a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and let $\mathbf{x}_j$ be a convex combination of $\mathbf{y}_{j1}, \ldots, \mathbf{y}_{jm_j}$, $j = 1, \ldots, m$. Show that $\mathbf{x}$ is a convex combination of $\mathbf{z}_1, \ldots, \mathbf{z}_p$, which are the distinct elements of the set $\{\mathbf{y}_{jk} : k = 1, \ldots, m_j, j = 1, \ldots, m\}$.

9. Let S be any subset of E^n. The set $\hat{S}$ of all convex combinations of points of S is the *convex hull* of S.
(a) Using Problem 8, show that $\hat{S}$ is convex.
(b) Using Proposition 1.6, show that if K is convex and $S \subset K$, then $\hat{S} \subset K$. Thus the convex hull is the smallest convex set containing S.

10. Given $\mathbf{x}_0$ and $\delta > 0$, let $C = \{\mathbf{x} : |x^i - x_0^i| \leq \delta,\ i = 1, \ldots, n\}$, an *n-cube* with center $\mathbf{x}_0$ and side length 2δ. The *vertices* of C are those $\mathbf{x}$ with $|x^i - x_0^i| = \delta$ for $i = 1, \ldots, n$. Show that C is the convex hull of its set of vertices. [*Hint*: Use induction on n.]

11. Let K be a closed subset of E^n such that both K and its complement $E^n - K$ are nonempty convex sets. Prove that K is a half-space.

12. Let A and B be convex subsets of E^n. The *join* of A and B is the set of all $\mathbf{x}$ such that $\mathbf{x}$ lies on a line segment with one endpoint in A and the other in B. Show that the join of A and B is a convex set.

for every $p \in S$. If ϕ is a constant function, $\phi(p) = c$ for every $p \in S$, then we write cf instead of ϕf.

Restriction of a function

Often one is interested only in the values of a function f for elements of some subset A of its domain. The *restriction* of f to A is the function with domain A and the same values as f there. It is denoted by $f \mid A$. Thus

$$f \mid A = \{(p, f(p)) : p \in A\}.$$

For instance, if a real-valued function f is integrated over an interval $I \subset E^1$, then it is only $f \mid I$ which is important. The values of f outside I do not affect the integral.

Images, inverse images

Let f be a function from a set S into a set T. The *image* under f of a set $A \subset S$ is the set $f(A) = \{f(p) : p \in A\}$. It is a subset of T, and in fact the restriction $f \mid A$ is a function from A onto $f(A)$. The *inverse image* of a set $B \subset T$ is the set $f^{-1}(B) = \{p : f(p) \in B\}$. It is a subset of S.

EXAMPLE 1. Let $f(x) = x^2$. Then $f([-2, 2]) = [0, 4]$, $f(E^1) = [0, \infty)$, $f^{-1}([1, 3]) = [-\sqrt{3}, -1] \cup [1, \sqrt{3}]$. The function f is not univalent since $f(-x) = f(x)$.

EXAMPLE 2. Let f be a function from a set S into a set T. Show that

$$(*) \qquad A \subset f^{-1}(f(A))$$

for any $A \subset S$. Consider any $p \in A$. Then $f(p) \in f(A)$ by definition of $f(A)$. Take $B = f(A)$ in the definition of inverse image set above. Then $p \in f^{-1}(f(A))$. Since this is true for each $p \in A$, we get (*).

EXAMPLE 3. Show that if f is univalent in Example 2, then

$$(**) \qquad A = f^{-1}(f(A)).$$

It suffices to show that $A \supset f^{-1}(f(A))$, since the opposite inclusion is (*). Consider any $p \in f^{-1}(f(A))$. Then $f(p) \in f(A)$. Therefore $f(p) = f(p')$ for some $p' \in A$. Since f is univalent, $p = p'$. Thus $p \in A$ as required.

PROBLEMS

1. (a) Let $f(x) = \cos x$. Find $f(E^1)$, $f([-\pi/4, \pi/2])$, $f^{-1}([0, 1])$.
(b) Let $g = f \mid [0, \pi]$. Find $g([0, \pi])$, $g^{-1}([0, 1])$. Is g univalent?

2. The equations $s = (x^2 + y^2)^{1/2}$, $t = x - y$ define a transformation $\mathbf{f}$ from E^2 into E^2, such that $\mathbf{f}(x, y) = (s, t)$. Let $A = \{(x, y) : x^2 + y^2 \le a^2\}$, where $a > 0$ is given.
(a) Find $\mathbf{f}(A)$.
(b) Find $\mathbf{f}^{-1}(A)$.

3. Let f be a function from S into T. Show that, for any $B \subset T$:
(a) $B \supset f(f^{-1}(B))$.
(b) $B = f(f^{-1}(B))$, if f is onto B.

4. Let f be a function from S into T. Show that, for any subsets A and B of S:
(a) $f(A \cup B) = f(A) \cup f(B)$.
(b) $f(A \cap B) \subset f(A) \cap f(B)$.
(c) If f is univalent, then $f(A \cap B) = f(A) \cap f(B)$.

5. Let f be a function from S into T. Show that, for any subsets C and D of T:
(a) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(b) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
(c) $f^{-1}(D^c) = [f^{-1}(D)]^c$.

6. Let S and T be sets, and let $\pi(p, q) = p$ for all $p \in S$, $q \in T$. The function π *projects* $S \times T$ onto S. Let $R \subset S \times T$ be a relation. Show that R is a function if and only if $\pi | R$ is univalent and onto S.

7. Let $1 \leq s \leq n - 1$. Let us regard E^n as the cartesian product $E^s \times E^{n-s}$, and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}' = (x^1, \ldots, x^s)$, $\mathbf{x}'' = (x^{s+1}, \ldots, x^n)$. Let $\pi(\mathbf{x}) = \mathbf{x}'$ be the projection of E^n onto E^s. Show that $\pi(A)$ is an open subset of E^s if A is an open subset of E^n.

8. Let $A \subset E^s$, $B \subset E^{n-s}$, and regard the cartesian product $A \times B$ as a subset of E^n, as in Problem 7.
(a) Show that $A \times B$ is open if both A and B are open.
(b) Show that $A \times B$ is closed if both A and B are closed.

2.2 Limits and continuity of transformations

Let us now suppose that $\mathbf{f}$ is a function from a set $D \subset E^n$ into E^m, where n and m are positive integers. As already mentioned, such functions are called *transformations* in this book.

The definition of "limit" for transformations is patterned after the one encountered in elementary calculus for real-valued functions of one variable. A *punctured* neighborhood of $\mathbf{x}_0$ is a neighborhood with the center $\mathbf{x}_0$ removed. Let us assume that D contains some punctured neighborhood of $\mathbf{x}_0$. For the definition of "limit," $\mathbf{x}_0$ itself need not be in D. If $\mathbf{x}_0 \in D$, the value of f at $\mathbf{x}_0$ is irrelevant.

Definition. If for every neighborhood V of $\mathbf{y}_0$ there is a punctured neighborhood U of $\mathbf{x}_0$ such that $\mathbf{f}(U) \subset V$, then $\mathbf{y}_0$ is the *limit* of the transformation $\mathbf{f}$ at $\mathbf{x}_0$ (Figure 2.1).

In the definition it is understood that the radius of U is small enough so that $U \subset D$. The notations $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) \to \mathbf{y}_0$ as $\mathbf{x} \to \mathbf{x}_0$ are used to mean that $\mathbf{y}_0$ is the limit of $\mathbf{f}$ at $\mathbf{x}_0$.

It is shown in Theorem 2.7 that the composite of two continuous transformations is continuous. In Section 4–4 it is shown that any differentiable transformation is continuous. For a real valued function f of one variable, differentiability of f at x_0 is equivalent to the existence of the derivative $f'(x_0)$.

Limits at ∞

Let us call a set of the form $\{\mathbf{x} : |\mathbf{x}| > b\}$ a *punctured neighborhood* of ∞. The definition of "limit at ∞" then reads: $\mathbf{y}_0 = \lim_{|\mathbf{x}| \to \infty} \mathbf{f}(\mathbf{x})$ if for every neighborhood V of $\mathbf{y}_0$ there exists a punctured neighborhood U of ∞ such that $\mathbf{f}(U) \subset V$.

When f is real valued we say that $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = +\infty$ if for every $C > 0$ there is a punctured neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) > C$ whenever $\mathbf{x} \in U$. The definition of "$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = -\infty$" is similar.

PROBLEMS

1. Find the limit at $\mathbf{x}_0$ if it exists.
(a) $f(x, y) = xy/(x^2 + y^2)$, $\mathbf{x}_0 = \mathbf{e}_1 + \mathbf{e}_2$.
(b) $f(x, y) = xy/(x^2 + y^2)$, $\mathbf{x}_0 = (0, 0)$.
(c) $f(x) = (1 - \cos x)/x^2$, $x_0 = 0$. [*Hint*: $\lim_{x \to 0} (\sin x)/x = 1$.]
(d) $\mathbf{f}(x) = |x - 2|\mathbf{e}_1 + |x + 2|\mathbf{e}_2$, $x_0 = 3$.
(e) $\mathbf{f}(x, y) = y\mathbf{e}_1 + (xy)^2/[(xy)^2 + (x - y)^2]\mathbf{e}_2$, $\mathbf{x}_0 = (0, 0)$.
At which points is each of these functions continuous?

2. Prove (2) of Proposition 2.1.

3. Show that if $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$, then $|\mathbf{y}_0| = \lim_{\mathbf{x} \to \mathbf{x}_0} |\mathbf{f}(\mathbf{x})|$. Prove that the converse holds if $\mathbf{y}_0 = \mathbf{0}$.

4. Let $f(x) = |x|^a$, where $a > 0$. Show that f is continuous at $x_0 = 0$ directly from the definition of continuous function.

5. Let $f(x, y) = x \cos(y^{-1})$ if $y \neq 0$, and $f(x, 0) = 0$. At which points is f continuous?

6. Find the limit if it exists.

(a) $\displaystyle \lim_{(x, y) \to (0, 0)} \frac{x^4 + y^4}{x^2 + y^2}$. (b) $\displaystyle \lim_{(x, y) \to (0, 0)} \frac{xy^2}{x^2 + y^4}$.

(c) $\displaystyle \lim_{|\mathbf{x}| \to \infty} \frac{(\mathbf{x} \cdot \mathbf{x}_1)(\mathbf{x} \cdot \mathbf{x}_2)}{\mathbf{x} \cdot \mathbf{x}}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given vectors not $\mathbf{0}$.

(d) $\displaystyle \lim_{|\mathbf{x}| \to \infty} \frac{|\mathbf{x} - \mathbf{x}_1|}{|\mathbf{x} - \mathbf{x}_2|}$.

7. (a) Let $g(\mathbf{x}) = |\mathbf{f}(\mathbf{x})|^a$ where $a > 0$. Suppose that $\mathbf{f}$ is continuous at $\mathbf{x}_0$ and $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. Show that g is continuous at $\mathbf{x}_0$.
(b) Use (a) and Problem 3 to give another proof that the function in Example 3 is continuous at (0, 0).

8. Let $y_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$, $z_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x})$. Show that if $z_0 \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{y_0}{z_0}.$$

9. Show that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = +\infty$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0}[f(\mathbf{x})]^{-1} = 0$ and $f(\mathbf{x}) > 0$ for every $\mathbf{x}$ in some punctured neighborhood of $\mathbf{x}_0$.

10. Prove Proposition 2.2 in two different ways. [*Hints*: For one proof use the definition of limit directly and the inequality, for any vector $\mathbf{h} = (h^1, \ldots, h^n)$,

$$|h^i| \leq |\mathbf{h}| \leq \sqrt{n}(|h^1| + \cdots + |h^n|).$$

Take $\mathbf{h} = \mathbf{f}(\mathbf{x}) - \mathbf{y}_0$. For the other proof, write $\mathbf{f} = f^1\mathbf{e}_1 + \cdots + f^n\mathbf{e}_n$ and note that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{e}_i = f^i(\mathbf{x})$.]

2.3 Sequences in E^n

An *infinite sequence* is a function whose domain is the set of positive integers. For brevity, we use the term "sequence" to mean infinite sequence. In this section let us consider sequences with values in E^n. It is customary to denote by $\mathbf{x}_m$ the value of the function at the integer $m = 1, 2, \ldots$, and to call $\mathbf{x}_m$ the mth *term* of the sequence. The sequence itself is denoted by $\mathbf{x}_1, \mathbf{x}_2, \ldots$, or for brevity by $[\mathbf{x}_m]$. It must not be confused with the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ whose elements are the terms of the sequence. This set may be finite or infinite. For instance if $x_m = (-1)^m$ then the sequence is $-1, 1, -1, \ldots$, and the set $\{x_1, x_2, \ldots\}$ has only two elements -1 and 1.

Definition. Suppose that for every $\varepsilon > 0$ there exists a positive integer N such that $|\mathbf{x}_m - \mathbf{x}_0| < \varepsilon$ for every $m \geq N$. Then $\mathbf{x}_0$ is the *limit* of the sequence $[\mathbf{x}_m]$.

The notations "$\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m$" and "$\mathbf{x}_m \to \mathbf{x}_0$ as $m \to \infty$" are used to mean that $\mathbf{x}_0$ is the limit of the sequence $[\mathbf{x}_m]$. A sequence is called *convergent* if it has a limit, otherwise *divergent*. The integer N in the definition depends of course on ε. Given ε there is a smallest possible choice for N. However, for purposes of the theory of limits it is of no interest to calculate it. What matters is the fact that some N exists.

Proposition 2.6. *Let* $\mathbf{x}_0 = \lim_{m\to\infty} \mathbf{x}_m, \mathbf{y}_0 = \lim_{m\to\infty} \mathbf{y}_m$. *Then*:

(a) $\mathbf{x}_0 + \mathbf{y}_0 = \lim_{m\to\infty}(\mathbf{x}_m + \mathbf{y}_m)$.

(b) $c\mathbf{x}_0 = \lim_{m\to\infty} c\mathbf{x}_m$ *for any scalar* c.

(c) $\mathbf{x}_0 \cdot \mathbf{y}_0 = \lim_{m\to\infty} \mathbf{x}_m \cdot \mathbf{y}_m$.

Let x_m^i denote the ith component of the vector $\mathbf{x}_m$.