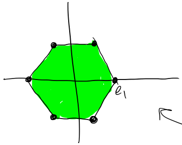
HW3 MTH 428.528 Spring 2025

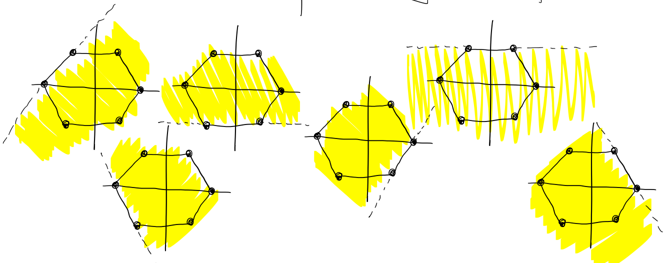
Friday, February 7, 2025 2:25 PM

§1.5 #1 Show closed + convex :

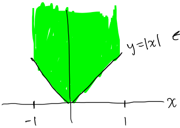
(a) hexagon w/ center at $(0,0)$ and one vertex at e_1



intersection of 6 half planes

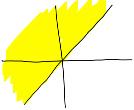


(b) $\{(x,y) : y \geq |x|, -1 \leq x \leq 1\}$

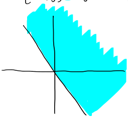


intersection of 2 half-planes

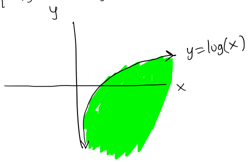
$\{(x,y) : y \geq x\}$



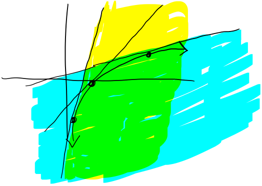
and $\{(x,y) : y \geq -x\}$



(c) $\{(x,y) : y \leq \log(x), x>0\}$

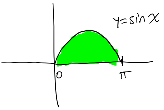


can think of this area as intersection of half-planes determined by lines tangent to $y=\log(x)$

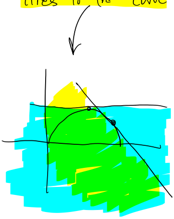


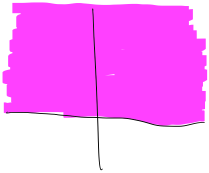
etc...

(d) $\{(x,y) : 0 \leq y \leq \sin(x), 0 \leq x \leq \pi\}$



intersection of half planes determined by tangent lines to the curve AND above $y=0$





etc

#3) Write $\frac{1}{4}e_1 + \frac{1}{2}e_2$ as convex combo of e_1 and $\frac{4}{3}e_2 - e_1$.

Also as convex combo of $0, e_2$, and $e_1 + e_2$.

Combo of $\frac{4}{3}e_2 - e_1$ and e_1

Goal: find weights c_1, c_2 so that $c_1 + c_2 = 1$ and

$$\underbrace{c_1 e_1 + c_2\left(\tfrac{4}{3}e_2 - e_1\right)} = \tfrac{1}{4}e_1 + \tfrac{1}{2}e_2$$

$$= (c_1 - c_2)e_1 + \tfrac{4}{3}c_2 e_2$$

So we have

$$\begin{cases} (c_1 - e_2)e_1 + \tfrac{4}{3}c_2 e_2 = \tfrac{1}{4}e_1 + \tfrac{1}{2}e_2 \\ c_1 + c_2 = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} c_1 - c_2 = \tfrac{1}{4} \\ \tfrac{4}{3}c_2 = \tfrac{1}{2} \longrightarrow \boxed{c_2 = \tfrac{3}{8}} \\ c_1 + c_2 = 1 \longrightarrow c_1 = 1 - c_2 = 1 - \tfrac{3}{8} = \tfrac{5}{8} \end{cases}$$

these obey also $c_1 - c_2 = \frac{1}{4}$ b/c $\frac{5}{8} - \frac{3}{8} = \frac{2}{8} = \frac{1}{4}$ ✓✓

Thus we have

$$\tfrac{5}{8}e_1 + \tfrac{3}{8}\left(\tfrac{4}{3}e_2 - e_1\right) = \tfrac{1}{4}e_1 + \tfrac{1}{2}e_2 \quad ✓$$

combo of $0, e_2, e_1 + e_2$

Goal: find weights c_1, c_2, c_3 so that $c_1 + c_2 + c_3 = 1$ and

$$c_1(0) + \underbrace{c_2 e_2 + c_3(e_1 + e_2)} = \tfrac{1}{4}e_1 + \tfrac{1}{2}e_2$$

$$(c_2 + c_3)e_2 + c_3 e_1$$

$$\Longrightarrow \begin{cases} c_1 + c_2 + c_3 = 1 \longrightarrow c_1 = 1 - c_2 - c_3 = 1 - \tfrac{1}{4} - \tfrac{1}{4} = \tfrac{1}{2} \\ c_2 + c_3 = \tfrac{1}{2} \longrightarrow c_2 + \tfrac{1}{4} = \tfrac{1}{2} \longrightarrow \boxed{c_2 = \tfrac{1}{4}} \\ c_3 = \tfrac{1}{4} \end{cases}$$

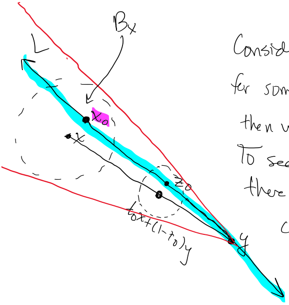
Thus

$$\tfrac{1}{2}(0) + \tfrac{1}{4}e_2 + \tfrac{1}{4}(e_1 + e_2) = \tfrac{1}{4}e_1 + \tfrac{1}{2}e_2 \quad ✓✓$$

#6 Let K be convex. Prove its interior and closure are convex.

Proof (interior): Assume K is convex, so for all $x, y \in K$, the line connecting x and y lies in K, i.e. all points of form $tx + (1-t)y$, $t \in [0,1]$ lies in K.

Let $x, y \in \text{int}(K)$, so $\exists\, \epsilon > 0$ s.t. the ball $B_x = \{z : \|x - z\| < \epsilon_x\} \subset K$.



$t_0 \in (0,1) \Rightarrow$ radius of this ball is smaller than radius of B_x

Consider open ball $B_{t_0} = \{z : \|z - (t_0x+(1-t_0)y\| < t_0\epsilon\}$ for some arbitrary $t_0 \in (0,1)$. Let $z_0 \in B_{t_0}$. If we argue that $z_0 \in K$ then we will have shown that $B_{t_0} \subset K$, since z_0 is arbitrary. To see it, consider line L connecting z_0 and y. By construction, there must be some $x_0 \in B_x$ s.t. the line connecting x_0 and y contains z_0. Since $x_0 \in B_x \subset K$, we know $x_0 \in K$, so the line segment connecting x_0 and y lies in K.

Thus in particular, $z_0 \in K$. Since $z_0 \in B_{t_0}$ was arbitrary, we get that $B_{t_0} \subset K$. Since $t_0 \in (0,1)$ was arbitrary, we see that all points on the line connecting x and y have a nhbd lying in K. Thus the line connecting x and y lies in $\mathrm{int}(K)$, completing the proof. ∎

Proof (closure): Let $x, y \in \mathrm{cl}(K)$. Since $x, y \in \mathrm{cl}(K) = K \cup \mathrm{fr}(K)$.

WLOG assume $x, y \in \mathrm{fr}(K)$ (so they are limit points!).

Thus $\exists$ sequences $\{x_n\} \subset K$ and $\{y_n\} \subset K$ s.t. $x_n \to x$ and $y_n \to y$.

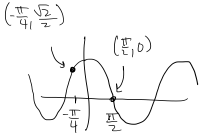
Consider line L connecting x and y given by $tx+(1-t)y$. We see that

$$tx+(1-t)y = \lim_{n\to\infty} tx_n+(1-t)y_n$$

In other words, all points $z_0 = t_0x+(1-t_0)y$ have a sequence $w_n = t_0x_n+(1-t_0)y_n \in K$ converging to z_0.

Since $w_n \in K$ for all n, we know $\lim_{n\to\infty} w_n \in \mathrm{cl}(K)$, which shows all points of L lie in $\mathrm{cl}(K)$, completing the proof. ∎

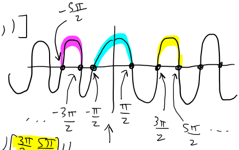
§2.1 #1 (a) Let $f(x) = \cos(x)$. Find $f(E^1)$, $f([-\frac{\pi}{4}, \frac{\pi}{2}])$, $f^{-1}([0,1])$.



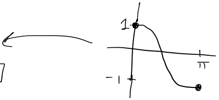
Soln: $f(E') = \{\cos(x) : x \in \mathbb{R}\} = [-1,1]$

$f([-\frac{\pi}{4}, \frac{\pi}{2}]) = [0, \frac{\sqrt{2}}{2}]$

$f^{-1}([0,1]) = \dots \cup [-\frac{5\pi}{2}, -\frac{3\pi}{2}] \cup [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}] \cup \dots$



(b) Let $g = f|_{[0,\pi]}$



$g([0,\pi]) = [-1,1]$

$g^{-1}([0,1]) = [0, \frac{\pi}{2}]$

yes g is univalent (i.e. one-to-one)

#2| $s = \sqrt{x^2+y^2}$, $t = x-y$ $f(x,y) = (s,t) = \left(\sqrt{x^2+y^2}, x-y\right)$

Let $A = \{(x,y) : x^2+y^2 \le a^2\}$ for some $a > 0$

(a) Find $f(A)$

Soln:

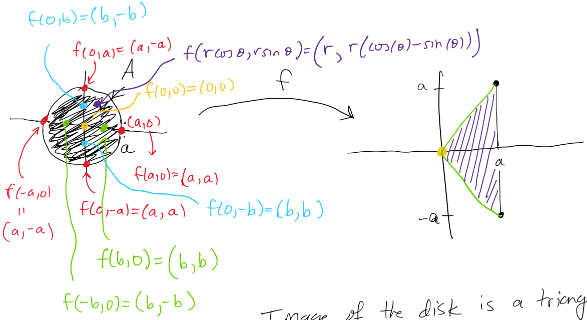


Image of the disk is a triangle!

#4| $f: S \to T$ and $A, B \subset S$

Show...

(a) $f(A \cup B) = f(A) \cup f(B)$

Proof: Let $z \in f(A \cup B)$, so $\exists\, w \in A \cup B$ s.t. $z = f(w)$.
Either $w \in A$ or $w \in B$, so we know either $f(w) \in f(A)$ or $f(w) \in f(B)$. Thus $z \in f(A) \cup f(B)$.
So we see $f(A \cup B) \subseteq f(A) \cup f(B)$.
For the converse, let $z \in f(A) \cup f(B)$. Then either $z \in f(A)$ or $z \in f(B)$, hence either $\exists\, w \in A$ s.t. $f(w) = z$ or $\exists\, s \in B$ s.t. $f(s) = W$. Thus

$$f(A) \cup f(B) \subseteq f(A \cup B).$$

Therefore $f(A \cup B] = f(A) \cup f(B)$. ■

(b) $f(A \cap B) \subseteq f(A) \cap f(B)$

Proof: Let $z \in f(A \cap B)$ so $\exists\, w \in A \cap B$ s.t. $f(w) = z$.
Therefore $w \in A$ and $w \in B$, so $z \in f(A) \cap f(B)$, completing the proof. ■

one-to-one (c) If f is univalent, then $f(A \cap B) = f(A) \cap f(B)$.

Proof: By (b) we know $f(A \cap B) \subseteq f(A) \cap f(B)$.
Let $z \in f(A) \cap f(B)$, so $z \in f(A)$ and $z \in f(B)$.
Since f is univalent, $\exists$ unique $w \in S$ s.t. $f(w) = z$.

Therefore $w \in A \cap B$, so $z \in f(A \cap B)$, completing the proof that $f(A) \cap f(B) \subseteq f(A \cap B)$. ■

§2.2 #1 Find limit if it exists:

(a) $$\lim_{(x,y) \to (1,1)} \frac{xy}{x^2 + y^2}$$

Soln: The function is continuous at $(1,1)$, so

$$\lim_{(x,y) \to (1,1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$$

(b) $$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$$

Soln: Limit DNE because...

along line $y = x$: $(x,y) = (x,x)$, so

$$\lim \quad \frac{x^2}{\quad} = \frac{1}{\quad}$$

$$(x,x)\to(0,0) \quad 2x^2 \qquad\qquad 2$$

while along x-axis:

$$\lim_{(x,0)\to(0,0)} \frac{(x)(0)}{x^2+0^2} = 0$$

(c) $$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{0}{0} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2}$$

(d) $$\lim_{x\to 3} |x-2|e_1 + |x+2|e_2 = \lim_{x\to 3} \left(|x-2|, |x+2|\right)$$

$$= \left(|3-2|, |3+2|\right)$$

$$= (1,5)$$

(e) $$\lim_{(x,y)\to(0,0)} ye_1 + \frac{(xy)^2}{(xy)^2+(x-y)^2} e_2$$

$$= \lim_{(x,y)\to(0,0)} \left(y, \underbrace{\frac{x^2y^2}{x^2y^2+(x-y)^2}}_{\text{along line } y=x \text{ we get}}\right)$$

$$\lim_{(x,x)\to(0,0)} \frac{x^4}{x^4+0^2} = 1$$

while along line $y=2x$, we get

$$\lim_{(x,x)\to(0,0)} \frac{4x^4}{4x^4+(-x)^2} = \lim_{x\to 0} \frac{4x^4}{4x^4+x^2} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{16x^3}{16x^3+2x}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{48x^2}{48x^2+2}$$

$$= \frac{0}{0+2} = 0$$

So limit DNE

#6

(a) $$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$$

Soln: Good way to do it is with polar coords : $\begin{cases} x=r\cos(\theta) \\ y=r\sin(\theta) \end{cases}$

$$\Rightarrow \lim_{\substack{(r,\theta)\to(0,\theta) \\ \uparrow}} \frac{r^4\cos^4(\theta)+r^4\sin^4(\theta)}{r^2(\cos^2(\theta)+\sin^2(\theta))}$$

letting θ be any value takes care of all possible approaches to $(0,0)$

$$= \lim_{(r,\theta)\to(0,\theta)} \frac{r^4(\cos^4(\theta)+\sin^4(\theta))}{r^2}$$

$\nearrow$ $\sin^2(\theta)+\cos^2(\theta)=1$

$$= \lim_{r\to 0} r^2(\cos^4(\theta)+\sin^4(\theta)) = 0$$

(b) $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

along parabola $x=y^2$ we have

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

but along x-axis $(x,y)=(x,0)$ we have

$$\lim_{x\to 0} \frac{0}{x^2} = 0$$

$\Rightarrow$ limit DNE

(c) $$\lim_{\|x\|\to\infty} \frac{(x\cdot x_1)(x\cdot x_2)}{x\cdot x} \quad ; x_1 x_2 \neq 0$$

Recall $x\cdot y = \|x\|\,\|y\|\cos(\theta)$, where θ is the angle between x and y.

So rewrite

$$x\cdot x_1 = \|x\|\,\|x_1\|\cos(\theta_{1,x}) \quad \swarrow \text{ angle changes as } x \text{ changes!}$$

$$x\cdot x_2 = \|x\|\,\|x_2\|\cos(\theta_{2,x})$$

Thus

$$\lim_{\|x\|\to\infty} \frac{(x\cdot x_1)(x\cdot x_2)}{x\cdot x} \underset{\substack{\uparrow \\ x\cdot x=\|x\|^2}}{=} \lim_{\|x\|\to\infty} \frac{\|x\|\left[\|x_1\|\cos(\theta_1) + \|x_2\|\cos(\theta_2)\right]}{\|x\|^2}$$

$$= \lim_{\|x\|\to\infty} \frac{\|x_1\|\cos(\theta_{1,x}) + \|x_2\|\cos(\theta_{2,x})}{\|x\|}$$

$$= 0$$

since $\underbrace{-1 \le \cos(\theta) \le 1}_{\text{always}}$, we see the numerator is bounded while the denominator grows to $+\infty$

(d) $$\lim_{\|x\|\to\infty} \frac{\|x-x_1\|}{\|x-x_2\|} = \lim_{\|x\|\to\infty} \frac{\|x-x_1\|}{\|x-x_2\|}$$

$$= \lim_{\|x\|\to\infty} \frac{\sqrt{(x-x_1)\cdot(x-x_1)}}{\sqrt{(x-x_2)\cdot(x-x_2)}}$$

$$= \lim_{\|x\|\to\infty} \sqrt{\frac{x\cdot x - 2x\cdot x_1 + x_1\cdot x_1}{x\cdot x - 2x\cdot x_2 + x_2\cdot x_2}}$$

$$= \lim_{\|x\|\to\infty} \sqrt{\frac{\|x\|^2 - 2x\cdot x_1 + \|x_1\|^2}{\|x\|^2 - 2x\cdot x_2 + \|x_2\|^2}}$$

$$= \lim_{\|x\|\to\infty} \sqrt{\frac{\|x\|^2 - 2\|x\|\|x_1\|\cos(\theta_1) + \|x_1\|^2}{\|x\|^2 - 2\|x\|\|x_2\|\cos(\theta_2) + \|x_2\|^2}}$$

think of as

$$= \lim_{w\to\infty} \sqrt{\frac{w^2 - 2w\|x_1\|\cos(\theta_1) + \|x_1\|^2}{w^2 - 2w\|x_2\|\cos(\theta_2) + \|x_2\|^2}}$$

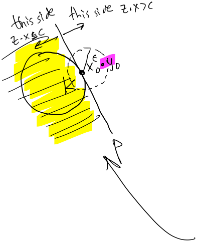
ratio of highest power of w

$$= \sqrt{1} = 1$$

Grad student problems

§1.5 #5 Prove supporting hyperplane of a closed convex set can contain no interior pt of K. $= \{x : z\cdot x = c\}$

Proof: Let K be convex and P be a supporting



hyperplane of K, so $P\cap K$ is nonempty and K is contained in one of the two closed half-spaces bounded by P. Say, we have $K\subseteq\{x: x\cdot z\leq c\}$.

Suppose $\exists\, x_0\in P$ s.t. x_0 is an interior point of K. We know that $x_0\cdot z=c$.

Then $\exists\,\epsilon>0$ s.t. the ball $B=\{x\in K: \|x-x_0\|<\epsilon\}\subset K$.

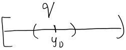
But this means $\exists\, y_0\in B$ s.t. $\|x_0-y_0\|<\epsilon$ and $y_0\cdot z>c$.

This contradicts that P is a supporting hyperplane, completing the proof. ■

§2.2 #4 Let $f(x)=|x|^a$ where $a>0$.

Show f is ctn at $x=0$ directly from the def.

Proof: We see $f(0)=0$, so for some $\epsilon>0$ let $\mathcal{V}=(-\epsilon,\epsilon)\subset\mathbb{R}$. Then $\mathcal{V}$ is an open set containing $f(0)$.



Goal: find open set $(-\delta,\delta)\subset\mathbb{R}$ s.t. $f((-\delta,\delta))\subset\mathcal{V}$.

Choose $\delta=\sqrt[a]{\epsilon}$, then

$$f(-\sqrt[a]{\epsilon},\sqrt[a]{\epsilon})=\{f(x)=|x|^a,\ x\in(-\sqrt[a]{\epsilon},\sqrt[a]{\epsilon})\}$$
$$=[0,\epsilon)\subset\mathcal{V},$$

thus f is continuous at $x=0$.