More generally, for any indexed collection of sets

$$\left(\bigcap_{\mu \in \mathscr{I}} A_\mu\right)^c = \bigcup_{\mu \in \mathscr{I}} A_\mu^c, \qquad \left(\bigcup_{\mu \in \mathscr{I}} A_\mu\right)^c = \bigcap_{\mu \in \mathscr{I}} A_\mu^c.$$

We have the following statement from Propositions 1.1 and 1.2.

Proposition 1.3. *The intersection of any indexed collection of closed sets is closed. The union of any finite indexed collection of closed sets is closed.*

Besides indexed collections, we sometimes consider unindexed collections of sets. (We use the term "collection of sets," rather than "set of sets," for a set whose elements are subsets of some given set S.) Let us use German script letters to denote collections of sets. For instance, the elements of a finite collection $\mathfrak{A} = \{A_1, \ldots, A_m\}$ of subsets of E^n are the sets $A_i \subset E^n$, $i = 1, \ldots, m$.

The *union* and *intersection* of a collection $\mathfrak{A}$ of sets are, respectively, the sets

$$\bigcup_{A \in \mathfrak{A}} A = \{p \in S : p \in A \text{ for some } A \in \mathfrak{A}\},$$

$$\bigcap_{A \in \mathfrak{A}} A = \{p \in S : p \in A \text{ for every } A \in \mathfrak{A}\}.$$

If each set of the collection is indexed by itself (taking $\mathscr{I} = \mathfrak{A}$, $A_A = A$), then this definition of union and intersection agrees with the one for indexed collections. Propositions 1.2 and 1.3 remain valid for unindexed collections.

Problems

1. Let U_1 be the δ-neighborhood of $\mathbf{x}_1$ and U_2 the δ-neighborhood of $\mathbf{x}_2$. Show that $U_1 \cap U_2$ is empty if and only if $\delta \le \frac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2|$.

2. Find int A, fr A, cl A if A is:
(a) $\{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| \le \delta\}$, $\delta > 0$.
(b) $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = \delta\}$, $\delta > 0$.
(c) $\{(x, y) : 0 < y < x + 1, x > -1\}$.
(d) $\{(r\cos\theta, r\sin\theta) : 0 < r < 1, 0 < \theta < 2\pi\}$.
(e) $\{(x, y) : x \text{ or } y \text{ is irrational}\}$.
(f) Any finite set.
(g) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $n = 1$.

3. In Problem 2 which sets are open? Which are closed?

4. Let A be any set. Show that int A is open, and that both fr A and cl A are closed.

5. (a) Show that the half space $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} < c\}$ is an open set.
[*Hint*: $|\mathbf{z} \cdot \mathbf{y} - \mathbf{z} \cdot \mathbf{x}| \le |\mathbf{z}||\mathbf{y} - \mathbf{x}|$.]
(b) Show that $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} \ge c\}$ is a closed set, using (a), and that the hyperplane $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = c\}$ is a closed set.

6. Show that:
(a) fr A = fr(A^c).
(b) cl A = cl(cl A).
(c) fr A = cl $A \cap$ cl(A^c).
(d) int A = (cl(A^c))^c.

7. Show by giving examples that the following are in general *false*:
(a) int(cl A) = int A.
(b) fr(fr A) = fr A.

8. Let A be open and B closed. Show that $A - B$ is open, and that $B - A$ is closed.

9. Show that:
(a) int($A \cap B$) = (int A) $\cap$ (int B).
(b) cl($A \cup B$) = (cl A) $\cup$ (cl B). [*Hint*: Part (a).]

10. Show that:
(a) int($A \cup B$) $\supset$ (int A) $\cup$ (int B).
(b) cl($A \cap B$) $\subset$ (cl A) $\cap$ (cl B).
Give examples in which = does not hold.

*1.5 Convex sets

In this section we give a brief introduction to the theory of convex sets. Our treatment of convexity continues in Section 3.6 with a discussion of convex and concave functions.

Proposition 1.4. *If $K_1, \ldots, K_m$ are convex sets, then their intersection $K_1 \cap \cdots \cap K_m$ is convex.*

Proof. Let $\mathbf{x}_1$, $\mathbf{x}_2$ be any two points of $K_1 \cap \cdots \cap K_m$, $\mathbf{x}_1 \neq \mathbf{x}_2$. Let l denote the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$. For each $j = 1, \ldots, m$, $\mathbf{x}_1, \mathbf{x}_2 \in K_j$. Since K_j is convex $l \subset K_j$ for each $j = 1, \ldots, m$. Thus $l \subset K_1 \cap \cdots \cap K_m$. □

In particular, Proposition 1.4 applies if each K_j is a half-space. A set that is the intersection of a finite number of closed half-spaces is called a *convex polytope*. Since a half-space is a convex set, any convex polytope is a convex set.

Example 1. Let T be a triangle in the plane E^2. Then T is the intersection of three half-planes, bounded by the lines through the sides of T.

A convex polytope is the set of all points $\mathbf{x}$ that satisfy a given finite system of linear inequalities of the form $\mathbf{z}^j \cdot \mathbf{x} \geq c^j$, $j = 1, \ldots, m$. The theory of linear programming is concerned with the problem of maximizing or minimizing a linear function subject to such a system of linear inequalities. It has various interesting economic and engineering applications [10, 13]. In Section 3.6 it is shown that the maximum and minimum values of a linear function must occur at "extreme points" of K, at least if K is compact.