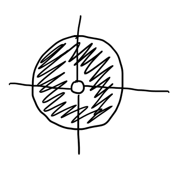
HW2 MTH 428/528 Spring 2025

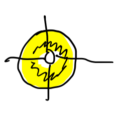
Wednesday, February 5, 2025 12:12 PM

§1.4 #2 Find int(A), fr(A), and cl(A) where...

(a) $A = \{x : 0 < \|x - x_0\| \le \delta\}, \delta > 0$

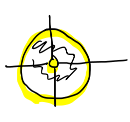
Soln: This is a closed spherical n-ball with its center point removed.



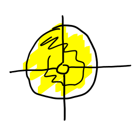


$\text{int}(A) = \{x : 0 < \|x - x_0\| < \delta\}$ ← open spherical n-ball with center removed

$\text{fr}(A) = \{x : \|x - x_0\| = \delta\} \cup \{x_0\}$ ← boundary of the sphere includes the "outside" but also the missing center point

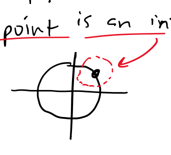


$\text{cl}(A) = \{x : \|x - x_0\| \le \delta\}$ ← fills center hole!



(b) $\{x : \|x - x_0\| = \delta\}, \delta > 0$ ← boundary of sphere!

Soln: $\text{int}(A) = \emptyset$ ← why empty? no point is an interior point!



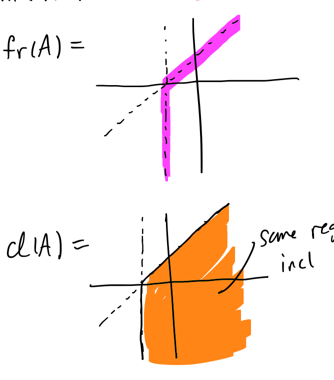
$\text{fr}(A) = A$ — all points are boundary points

$\text{cl}(A) = A$ — no limit points of the set except its own elements!

(c) $\{(x,y) : 0 < y < x+1, \; x > -1\}$

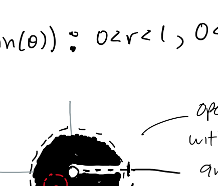
Soln:

$y = x+1$

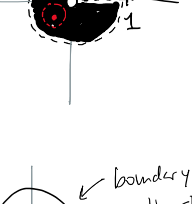


$\text{int}(A) = A$ — all pts are interior pts

$\text{fr}(A) =$



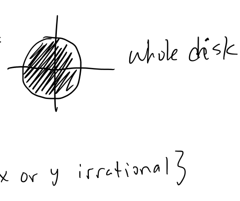
$\text{cl}(A) =$



same region incl boundary

(d) $\{(r\cos(\theta), r\sin(\theta)) : 0 < r < 1, \; 0 < \theta < 2\pi\}$

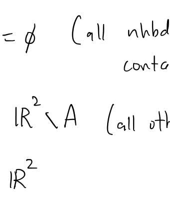
Soln:



open disk with a slit and no center

$\text{int}(A) = A$

$\text{fr}(A) =$



boundary of circle with slit and center pt

$\text{cl}(A) =$

whole disk

(e) $\{(x,y) : x \text{ or } y \text{ irrational}\}$

$\text{int}(A) = \emptyset$ (all nhbd of points there contain points outside)

uses idea that all open sets in $\mathbb{R}^1$ contain both rat'l and irrat'l pts

$\text{fr}(A) = \mathbb{R}^2 \setminus A$ (all other pts of plane)

$\text{cl}(A) = \mathbb{R}^2$

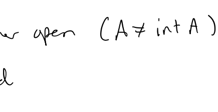
(f) finite set $\{x_1, \dots, x_n\}$

$\text{int}(A) = \emptyset$ (all nhbd contain pts outside of A)

$\text{fr}(A) = A$ (same reason)

$\text{cl}(A) = A \cup \text{fr}(A) = A$

(g) $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset E^1$



$\text{int}(A) = \emptyset$ (same as)

$\text{fr}(A) = 0$ (all nhbd of 0 contains pts inside and outside of A)

$\text{cl}(A) = A \cup \{0\}$

#3 Which sets above are open? closed?

(a) neither open ($A \ne \text{int} A$) nor closed ($A \ne \text{cl}(A)$)

(b) closed

(c) open

(d) open

(e) neither open ($A \ne \text{int} A$) nor closed ($A \ne \text{cl}(A)$)

(f) closed

(g) neither open ($A \ne \text{int} A$) nor closed ($A \ne \text{cl}(A)$)

#5 (a) Show $\{x : z \cdot x < c\}$ is open ← call this set W

Soln: Let $y \in W$ so we know $z \cdot y < c$

> Goal: Show $\exists \delta > 0$ so that
> $\mathcal{U} := \{x : \|x - y\| < \delta\} \subset W$

Choose $\delta = \frac{c - |z \cdot y|}{\|z\|}$, and let $u \in \mathcal{U}$.

Then we know $\|u - y\| < \delta = \frac{c - |z \cdot y|}{\|z\|}$

$-z \cdot y > -c$

$c - z \cdot y > 0$

Now compute

$$|z \cdot u| = |z \cdot (u - y + y)|$$
$$= |z \cdot (u - y) + z \cdot y|$$

Δ-ineq $\le |z \cdot (u - y)| + |z \cdot y|$

Cauchy inequality (given hint) $\le \|z\| \, \|u - y\| + |z \cdot y|$

$$< \|z\| \left(\frac{c - |z \cdot y|}{\|z\|} \right) + |z \cdot y|$$
$$= c - |z \cdot y| + |z \cdot y|$$
$$= c$$

$\|z\|\delta + \tilde{c} = c$

$\delta = \frac{c - \tilde{c}}{\|z\|}$

Thus we see that $u \in W$, completing the proof. ∎

(b) Show that $\{x : x \cdot z \ge c\}$ is closed.

Proof: Complement of an open set is a closed set, and

$$\underbrace{\{x : x \cdot z < c\}^c}_{\text{complement of open set}} = \{x : x \cdot z \ge c\}$$

is closed. ∎

Graduate student problems

§1.4 #6 Show that...

(a) $\text{fr}(A) = \text{fr}(A^c)$

Proof: Given any $x \in \text{fr}(A)$, any nhbd U of x, intersects both A and A^c.

So $x \in \text{fr}(A^c)$.

The same argument shows any $x \in \text{fr}(A^c)$ is also in $\text{fr}(A)$. ∎

(b) $\text{cl}(A) = \text{cl}(\text{cl}(A))$

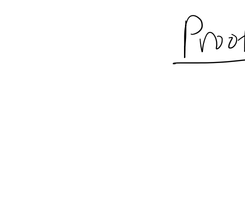
Proof: By def $\text{cl}(A) = A \cup \text{fr}(A)$ and $\text{cl}(\text{cl}(A)) = \text{cl}(A) \cup \text{fr}(\text{cl}(A))$.

So we see that $\text{cl}(A) \subset \text{cl}(\text{cl}(A))$.

Now suppose $x \in \text{cl}(\text{cl}(A)) = \text{cl}(A) \cup \text{fr}(\text{cl}(A))$.

If $x \in \text{cl}(A)$, then nothing to prove.

If $x \in \text{fr}(\text{cl}(A))$, then it means for any nhbd U of x, $U \cap \text{cl}(A) \ne \emptyset$ and $U \cap \text{cl}(A)^c \ne \emptyset$.



Let $z \in \mathcal{U} \cap \text{cl}(A)$. Then $\mathcal{U}$ is open set containing z which must intersect both A and A^c.

Thus $x \in \text{cl}(A)$, since any nhbd $\mathcal{U}$ of it contains a point in A and a pt in A^c.

Thus $\text{cl}(\text{cl}(A)) \subset \text{cl}(A)$, completing the proof. ∎

(c) $\text{fr}(A) = \text{cl}(A) \cap \text{cl}(A^c)$

Proof: Let $x \in \text{fr}(A)$, so $\forall$ open $\mathcal{U}$ with $x \in \mathcal{U}$ intersects both A and A^c.

But this means $x \in \text{fr}(A) \subset \text{cl}(A)$ and $x \in \text{fr}(A^c) \subset \text{cl}(A^c)$,

completing the proof. ∎

(d) $\text{int}(A) = (\text{cl}(A^c))^c$

Proof: We will show $\text{int}(A)^c = \text{cl}(A^c)$.

Let $x \in \text{int}(A)^c$, so $x \notin \text{int}(A)$ so for any open $\mathcal{U}$ with $x \in \mathcal{U}$, we have $\mathcal{U} \cap A^c \ne \emptyset$. But this means $x \in A^c \subset \text{cl}(A^c)$.

On the other hand, let $x \in \text{cl}(A^c)$, so for any open $\mathcal{U}$ with $x \in \mathcal{U}$, we know $\mathcal{U} \cap A^c \ne \emptyset$. This means we can't have $x \in \text{int}(A)$, thus $x \in \text{int}(A)^c$.

So we proved

$$\text{int}(A)^c = \text{cl}(A^c)$$

Taking complement of both sides gives

$$(\text{int}(A)^c)^c = \text{int}(A) = \text{cl}(A^c)^c,$$

completing the proof. ∎