Taking the inner product of each side with $\mathbf{v}_i$ and using the formula $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$, we obtain

$$\mathbf{x} \cdot \mathbf{v}_i = c^i.$$

The coefficients c^i (1.6) are just the components of $\mathbf{x}$ with respect to the orthonormal basis vectors.

Problems

1. Let $n = 4$, $\mathbf{x} = \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_4 = (1, -1, 0, 2)$, $\mathbf{y} = 3\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 = (3, -1, 1, 1)$. Find $\mathbf{x} + \mathbf{y}$, $\mathbf{x} - \mathbf{y}$, $|\mathbf{x} + \mathbf{y}|$, $|\mathbf{x} - \mathbf{y}|$, $|\mathbf{x}|$, $|\mathbf{y}|$, $\mathbf{x} \cdot \mathbf{y}$. Verify (1.1) and (1.2) in this example.

2. Prove that the standard euclidean inner product in E^n has the following four properties:
   (a) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. (b) $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$.
   (c) $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y})$. (d) $\mathbf{x} \cdot \mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$.

3. Using Problem 2, show that

$$(\mathbf{w} + c\mathbf{x}) \cdot (\mathbf{y} + d\mathbf{z}) = \mathbf{w} \cdot \mathbf{y} + c\mathbf{x} \cdot \mathbf{y} + d\mathbf{w} \cdot \mathbf{z} + cd\mathbf{x} \cdot \mathbf{z}.$$

4. Show that $\sum_{i=1}^{n} |x^i| \leq \sqrt{n}\,|\mathbf{x}|$, for any $\mathbf{x} = (x^1, \ldots, x^n)$. [*Hint*: First suppose that $x^i \geq 0$. Use Equation (1.1) with $y^i = 1$.]

5. Show that $2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 = |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2$. What does this say about parallelograms (see Figure 1.3)?

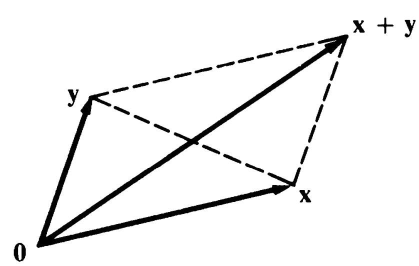

Figure 1.3

6. Show that $|\mathbf{x} + \mathbf{y}|\,|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x}|^2 + |\mathbf{y}|^2$ with equality if and only if $\mathbf{x} \cdot \mathbf{y} = 0$. What does this say about parallelograms?

7. Prove (1.3), using (1.2) and induction on m.

8. Let $n = 4$, and

$$\mathbf{v}_1 = \tfrac{1}{5}(3\mathbf{e}_1 + 4\mathbf{e}_3),$$
$$\mathbf{v}_2 = \tfrac{1}{5}(4\mathbf{e}_2 - 3\mathbf{e}_4),$$
$$\mathbf{v}_3 = (\sqrt{2}/10)(-4\mathbf{e}_1 + 3\mathbf{e}_2 + 3\mathbf{e}_3 + 4\mathbf{e}_4).$$

   Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal unit vectors. Find a unit vector $\mathbf{v}_4$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form an orthonormal basis for E^4.

9. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for E^n, and let

$$C = \left\{\mathbf{x} : \mathbf{x} = \sum_{i=1}^{n} t^i \mathbf{v}_i,\ 0 \le t^i \le 1 \quad \text{for} \quad i = 1, \ldots, n\right\}.$$

The set C is an *n-cube*. If each $t^i = 0$ or 1, $\mathbf{x}$ is called a *vertex* of C. What are the possible distances between vertices of C?

10. (*Gram–Schmidt process.*) Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis for E^n. Let $\mathbf{v}_1 = |\mathbf{x}_1|^{-1}\mathbf{x}_1$, $\mathbf{y}_2 = \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{v}_1)\mathbf{v}_1$, $\mathbf{v}_2 = |\mathbf{y}_2|^{-1}\mathbf{y}_2$, $\mathbf{y}_3 = \mathbf{x}_3 - (\mathbf{x}_3 \cdot \mathbf{v}_1)\mathbf{v}_1 - (\mathbf{x}_3 \cdot \mathbf{v}_2)\mathbf{v}_2$, $\mathbf{v}_3 = |\mathbf{y}_3|^{-1}\mathbf{y}_3, \ldots, \mathbf{v}_n = |\mathbf{y}_n|^{-1}\mathbf{y}_n$. Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for E^n.

11. Let $\mathscr{V}$ be a vector subspace of E^n, of dimension k; and consider its *orthogonal complement*

$$\mathscr{V}^\perp = \{\mathbf{y} : \mathbf{y} \cdot \mathbf{x} = 0 \quad \text{for all} \quad \mathbf{x} \in \mathscr{V}\}.$$

(a) Find an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for E^n, such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\mathscr{V}$ and $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is a basis for $\mathscr{V}^\perp$. [*Hint*: Apply Problem 10 to a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis for $\mathscr{V}$.]

(b) Show that each $\mathbf{x} \in E^n$ can be written in one and only one way as $\mathbf{x} = \mathbf{y} + \mathbf{z}$ with $\mathbf{y} \in \mathscr{V}$ and $\mathbf{z} \in \mathscr{V}^\perp$.

1.3 Elementary geometry of E^n

Such concepts as lines, planes, circles, and spheres in E^2 or E^3 have analogs in E^n for any dimension n. Let us begin with the concept of line in E^n.

Definition. Let $\mathbf{x}_1, \mathbf{x}_2 \in E^n$ with $\mathbf{x}_1 \neq \mathbf{x}_2$. The *line* through $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\{\mathbf{x} : \mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2, t \text{ any scalar}\}.$$

If we set $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$, then this can be rewritten as

$$\{\mathbf{x} : \mathbf{x} = \mathbf{x}_2 + t\mathbf{z}, t \text{ any scalar}\}.$$

In the plane E^2 the vector equation $\mathbf{x} = \mathbf{x}_2 + t\mathbf{z}$ becomes

$$x = x_2 + t(x_1 - x_2), \qquad y = y_2 + t(y_1 - y_2),$$

which, in elementary analytic geometry, are called "parametric equations" of the line through (x_1, y_1) and (x_2, y_2).

The *line segment* joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\{\mathbf{x} : \mathbf{x} = t\mathbf{x}_1 + (1 - t)\mathbf{x}_2, t \in [0, 1]\},$$

where $[a, b]$ denotes the set of real numbers t such that $a \le t \le b$ (Section 1.1).

For example, if $t = \frac{1}{2}$, then $\mathbf{x}$ is the midpoint of the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ (Figure 1.4). The points corresponding to $t = \frac{1}{3}, \frac{2}{3}$ trisect the line segment.

Problems

1. Let $n = 3$. Find the plane that contains the three points $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3 - 3\mathbf{e}_1$. Sketch its intersection with the first octant in E^3.

2. (a) Find the hyperplane in E^4 containing the four points $\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$, $3\mathbf{e}_4 - \mathbf{e}_2$.
   (b) Find the value of t for which $t(\mathbf{e}_1 - \mathbf{e}_2) + (1 - t)\mathbf{e}_4$ is in this hyperplane.

3. Let l denote the line in E^4 through $\mathbf{e}_1 - \mathbf{e}_3$ and $-\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4$. Find the hyperplane P through $\mathbf{e}_1 - \mathbf{e}_3$ to which l is perpendicular.

4. Let $\mathscr{V} = \{(x, y, z): 2x + 3y - z = 0\}$. Show that $\mathscr{V}$ is a 2-dimensional vector subspace of E^3, and find a basis for $\mathscr{V}$. ($\mathscr{V}$ is a vector subspace of E^n if $\mathbf{x}, \mathbf{y} \in \mathscr{V}$ imply $\mathbf{x} + \mathbf{y} \in \mathscr{V}$ and $c\mathbf{x} \in \mathscr{V}$ for any scalar c.)

5. Let $\mathscr{V} = \{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} = 0\}$, where $\mathbf{z} \neq \mathbf{0}$ is given. Show that $\mathscr{V}$ is an $(n - 1)$-dimensional vector subspace of E^n, and find a basis for $\mathscr{V}$.

6. Show that $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_1| = |\mathbf{x} - \mathbf{x}_2|\}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given points in E^n, is a hyperplane.

7. Show that $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_1| = c|\mathbf{x} - \mathbf{x}_2|\}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are given points in E^n and $0 < c < 1$, is an $(n - 1)$-sphere.

8. Let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n-1}$ be such that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_{n-1} - \mathbf{x}_0$ are linearly independent. Prove that there is exactly one hyperplane containing $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n-1}$.

9. A set $P = \{\mathbf{x} : \mathbf{z}_i \cdot \mathbf{x} = c_i \text{ for } i = 1, \dots, n - k\}$, where $\mathbf{z}_1, \dots, \mathbf{z}_{n-k}$ are linearly independent vectors, is called a *k-plane* in E^n. Let $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$ be such that $\mathbf{x}_1 - \mathbf{x}_0, \dots, \mathbf{x}_k - \mathbf{x}_0$ are linearly independent. Prove that there is exactly one k-plane containing $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$.

10. Prove that any line in E^n is a convex set.

11. Show that K is a convex set by directly applying the definition. Sketch K in the cases $n = 1, 2, 3$.
    (a) $K = \{\mathbf{x} : |x^1| + \cdots + |x^n| \leq 1\}$.
    (b) $K = \{\mathbf{x} = c^1\mathbf{v}_1 + \cdots + c^n\mathbf{v}_n,\ 0 \leq c^i \leq 1 \text{ for } i = 1, \dots, n\}$, where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for E^n. This is the *n-parallelepiped* spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n$ with $\mathbf{0}$ as a vertex.

12. Let P be a hyperplane. Prove that the line through any two points of P is contained in P. Why does this imply that P is a convex set?

1.4 Basic topological notions in E^n

We now introduce some basic concepts that are essential to a careful treatment of several-variable calculus. These concepts are developed further in Chapter 2.

Let us begin by making precise the idea of being "strictly inside" a set A, "strictly outside" A, or neither. Points with these properties will be called, respectively, interior, exterior, or frontier points. We first define the concept of neighborhood.