HW1 MTH 428/528 Spring 2025

Wednesday, January 29, 2025 11:16 AM

$$\boxed{\begin{array}{ll} (1.1) & |x\cdot y| \le \|x\|\,\|y\| \\ (1.2) & \|x+y\| \le \|x\| + \|y\| \end{array}}$$

§1.2 **#1** $n=4$, $x = e_1 - e_2 + 2e_4 = (1,-1,0,2)$

$y = 3e_1 - e_2 + e_3 + e_4 = (3,-1,1,1)$

$$x+y = (1,-1,0,2)+(3,-1,1,1) = (1+3,-1+(-1),0+1,2+1) = (4,-2,1,3)$$

$$x-y = (1-3,-1-(-1),0-1,2-1) = (-2,-2,-1,1)$$

$$\|x+y\| = \|(4,-2,1,3)\| = \sqrt{4^2+(-2)^2+1^2+3^2} = \sqrt{16+4+1+9} = \sqrt{30}$$

$$\|x-y\| = \sqrt{4+4+1+1} = \sqrt{10}$$

$$\|x\| = \sqrt{1+1+0+4} = \sqrt{6}$$

$$\|y\| = \sqrt{9+1+1+1} = \sqrt{12}$$

$$\begin{aligned} x\cdot y = (1,-1,0,2)\cdot(3,-1,1,1) &= (1)(3)+(-1)(-1)+(0)(1)+(2)(1) \\ &= 3+1+0+2 \\ &= 6 \end{aligned}$$

Verify (1.1): $|x\cdot y| = 6$

$\|x\|\,\|y\| = (\sqrt{6})(\sqrt{12}) = 6\sqrt{2}$

So, $|x\cdot y| \le \|x\|\,\|y\|$ ✓

verify (1.2):

$\|x+y\| = \sqrt{10} \approx 3.16$

$\|x\| + \|y\| = \sqrt{6} + \sqrt{12} \approx 5.91$

So, $\|x+y\| \le \|x\| + \|y\|$ ✓

#2 Prove... for $x = (x^1,\dots,x^n)$, $y = (y^1,\dots,y^n)$, $z = (z^1,\dots,z^n)$, $c \in \mathbb{R}$

(a) $x\cdot y = y\cdot x$

Proof: $x\cdot y \overset{\text{def}}{=} \sum_{k=1}^{n} x^k y^k \underset{\substack{\text{mult of} \\ \text{real \#'s is} \\ \text{commutative}}}{=} \sum_{k=1}^{n} y^k x^k \overset{\text{def}}{=} y\cdot x$

(b) $(x+y)\cdot z = x\cdot z + y\cdot z$

Proof: $(x+y)\cdot z = \sum_{k=1}^{n} (x^k+y^k) z^k = \sum_{k=1}^{n} x^k z^k + \sum_{k=1}^{n} y^k z^k = x\cdot z + y\cdot z$

(c) $(cx)\cdot y = c(x\cdot y)$

Proof: $(cx)\cdot y = \sum_{k=1}^{n} (cx^k) y^k = c \sum_{k=1}^{n} x^k y^k = c(x\cdot y)$

(d) $x\cdot x>0$ if $x\neq 0$

Proof: Suppose at least one of $x^1,\dots,x^n$ is nonzero. Then,

$$x\cdot x=\sum_{k=1}^{n}x^kx^k=\sum_{k=1}^{n}\underbrace{(x^k)^2}_{\substack{\uparrow\\ \text{each of}\\ \text{these}\geq 0,\\ \text{at least one is }>0}}>0$$

#3 Prove

$$(w+cx)\cdot(y+dz)=w\cdot y+c(x\cdot y)+d(w\cdot z)+cd(x\cdot z)$$

Proof: Using parts from #2:

$$(w+cx)\cdot(y+dz)\overset{(b)}{=}w\cdot(y+dz)+(cx)\cdot(y+dz)$$

$$\overset{(a)}{=}(y+dz)\cdot w+(y+dz)\cdot(cx)$$

$$\overset{(b)}{=}y\cdot w+(dz)\cdot w+y\cdot(cx)+(dz)\cdot(cx)$$

$$\overset{(c)}{=}y\cdot w+d(z\cdot w)+c(y\cdot x)+dc(z\cdot x)$$

$$\boxed{\text{Props of adding real numbers and (a)}}\hookrightarrow = w\cdot y+c(x\cdot y)+d(w\cdot z)+cd(x\cdot z)$$

§1.3 #1 Let $n=3$. Find the plane that contains e_1, e_2, and e_3-3e_1. Sketch its intersection with the first $\underbrace{\text{octant}}_{\substack{x^1\geq 0\\ x^2\geq 0\\ x^3\geq 0}}$ of E^3

Soln: plane has formula $\{x\in E^3 : x\cdot z=c\}$ ← unknowns

We know $\underset{(1,0,0)}{\underset{\|}{e_1}}, \overset{(0,1,0)}{\overset{\|}{e_2}}$, and $\underset{(-3,0,1)}{\underset{\|}{e_3-3e_1}}$ on the plane. Let $z=(z^1,z^2,z^3)$.

So,

$$e_1\cdot z=(1,0,0)\cdot(z^1,z^2,z^3)=z^1=c$$

$$e_2\cdot z=(0,1,0)\cdot z=z^2=c$$

$$(e_3-3e_1)\cdot z=(-3,0,1)\cdot(z^1,z^2,z^3)=-3z^1+z^3=c$$

Thus we have system

$$\begin{cases} z^1=c\\ z^2=c\\ -3z^1+z^3=c\end{cases}\longrightarrow \begin{aligned}-3c+z^3&=c\\ z^3&=4c\end{aligned}$$

Thus $z=(c,c,4c)$ with arbitrary $\underset{(c\neq 0)}{\underset{\uparrow}{c}}$. So choose, say, $c=1$,

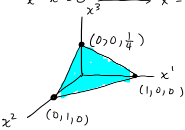
and we get $z=(1,1,4)$ and our plane is

$$\{x\in E^3 : x\cdot z=c\}=\{x\in E^3 : x^1+x^2+4x^3=1\}$$

Sketch: if $x^1 = x^2 = 0$ we get $4x^3 = 1 \to x^3 = \frac{1}{4} \to (0,0,\frac{1}{4})$ on the plane

$x^2 = x^3 = 0 \longrightarrow x_1 = 1 \longrightarrow (1,0,0)$ on the plane

$x^1 = x^3 = 0 \longrightarrow x^2 = 1 \longrightarrow (0,1,0)$ on the plane



#2 (a) Find hyperplane in E^4 containing $0, e_1+e_2, e_1-e_2+2e_3, 3e_4-e_2$

Soln (a): The plane has form $\{x \in E^4 : z \cdot x = c\}$

unknowns

Compute for $z = (z^1, z^2, z^3, z^4)$,

$0 \cdot z = 0 \longleftarrow$ useless!

$(e_1+e_2)\cdot z = z^1 + z^2 = c$

$(e_1 - e_2 + 2e^3)\cdot z = z^1 - z^2 + 2z^3 = c$

$(3e_4 - e_2)\cdot z = 3z^4 - z^2 = c$

So we get

$$\begin{cases} z^1 + z^2 = c \longrightarrow \boxed{z_1 = c - z^2} \\ z^1 - z^2 + 2z^3 = c \longrightarrow (\cancel{c} - z^2) - z^2 + 2z^3 = \cancel{c} \\ 3z^4 - z^2 = c \end{cases}$$

$-2z^2 + 2z^3 = 0$

$\boxed{z^3 = z^2}$

$\boxed{z^4 = \frac{z^2 + c}{3}}$

So the only constraints we got are

$$\begin{cases} z_1 = c - z^2 \\ z^3 = z^2 \\ z^4 = \frac{z^2 + c}{3} \end{cases}$$

free variables z^2 and c!

So take $z^2 = c = 1$ to get $\begin{cases} z^1 = 0 \\ z^3 = 1 \\ z^4 = 2/3 \end{cases}$

So our $z = (0, 1, 1, 2/3)$ and $c = 1$

Giving us plane

$$\{x \in E^4 : x \cdot z = c\} = \{x \in E^4 : x^2 + x^3 + \tfrac{2}{3}x^4 = 1\}$$

Part (b): Find t for which $t(e_1 - e_2) + (1-t)e_4$ is in the hyperplane.

Soln (b): We want

$t(e_1 - e_2) + (1-t)e_4 = (t, -t, 0, 1-t)$ lies in the plane

So compute

$(t, -t, 0, 1-t)\cdot z = c$

$$(t, -t, 0, 1-t)\cdot(0, 1, 1, \tfrac{2}{3}) = 1$$

$$0t - t + 0 + \tfrac{2}{3}(1-t) = 1$$

$$-t + \tfrac{2}{3} - \tfrac{2}{3}t = 1$$

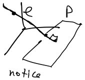
$$-\tfrac{5}{3}t = \tfrac{1}{3} \longrightarrow t = -\tfrac{1}{5}$$

note: your answer may be different if you choose different z^2 and c in part (a)

#3 Let ℓ denote line in E^4 thru $e_1 - e_3$ and $-e_1 + e_2 + 2e_4$

Find hyperplane P thru $e_1 - e_3$ to which ℓ is perpendicular.

in E^3...



So notice line ℓ

Soln: Recall line thru x_1 and x_2 is

$$\{x : x = t x_1 + (1-t) x_2\}$$

and plane obeys

unknowns

$$\{x : x \cdot z = c\}$$

Here we consider line thru $x_1 = e_1 - e_3 = (1, 0, -1, 0)$
$x_2 = -e_1 + e_2 + 2e_4 = (-1, 1, 0, 2)$

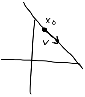
which is

$$\{x : x = t(1,0,-1,0) + (1-t)(-1,1,0,2)\}$$

$$= \{x : x = (t + (1-t), 1-t, -t, 2(1-t))\}$$

$$= \{x : x = (1, 1-t, -t, 2-2t)\}$$

$$= \{x : x = \underbrace{(1,1,0,2)}_{\text{"starting value"}} + t\,\underbrace{(0,-1,-1,-2)}_{\text{vector parallel to the line}}\}$$



— call it v

↑

need this to be perpendicular to the unknown plane

↑

ultimately means for the z in plane def that

$$\underbrace{v \cdot z = 0}_{\text{def of perpendicular}}$$

So our plane

- contains $e_1 - e_3$ and $-e_1 + e_2 + 2e_4$

$$(e_1 - e_3)\cdot z = z^1 - z^3 = c$$

$$(-e_1 + e_2 + 2e_4)\cdot z = -z^1 + z^2 + 2z^4 = c$$

- perpendicular part

$$v \cdot z = (0, -1, -1, -2)\cdot z = 0$$

$$-z^2 - z^3 - 2z^4 = 0$$

$\Longrightarrow$ we have system

$$\begin{cases} z^1 - z^3 = c \\ -z^1 + z^2 + 2z^4 = c \\ -z^2 - z^3 - 2z^4 = 0 \end{cases}$$

$$\longrightarrow \boxed{z^1 = c + z^3}$$

$$\longrightarrow -(c+z^3) + z^2 + 2z^4 = 0$$

$$\boxed{z^2 = c + z^3 - 2z^4}$$

$$-(c + z^3 - \cancel{2z^4}) - z^3 - \cancel{2z^4} = 0$$

$$-c - 2z^3 = 0$$

$$\boxed{z^3 = -\frac{c}{2}}$$

$$\boxed{z^2 = c - \frac{c}{2} - 2z^4 = \frac{c}{2} - 2z^4}$$

$$\boxed{z^1 = c - \frac{c}{2} = \frac{c}{2}}$$

We got

$$\Longrightarrow \begin{cases} z_1 = c/2 \\ z_2 = \frac{c}{2} - 2z^4 \\ z_3 = -c/2 \end{cases}$$

free!

so pick

$c = z^4 = 1$

$$\Longrightarrow \quad c = 1 \text{ and } z = \left(1/2, \tfrac{1}{2} - 2, -\tfrac{1}{2}, 1\right)$$
$$= \left(1/2, -3/2, -1/2, 1\right)$$

Thus the required plane is

$$\left\{ x \in E^4 : \tfrac{1}{2}x_1 - \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 + x^4 = 2 \right\}$$

#4 Let $\mathcal{V} = \{(x,y,z) : 2x + 3y - z = 0\}$

Show $\mathcal{V}$ is a 2-dim subspace of E^3

Soln: Let $a, b \in \mathcal{V}$ with $a = (a^1, a^2, a^3)$
$b = (b^1, b^2, b^3)$

Since $a, b \in \mathcal{V}$, we know

$$\begin{cases} 2a^1 + 3a^2 - a^3 = 0 \\ 2b^1 + 3b^2 - b^3 = 0 \end{cases}$$

Need to show $a + b \in \mathcal{V}$ and for any $c \in \mathbb{R}$, $ca \in \mathcal{V}$.

Compute

$$a + b = (a^1 + b^1, a^2 + b^2, a^3 + b^3)$$

Then, $2(a^1 + b^1) + 3(a^2 + b^2) - (a^3 + b^3)$

$$= (2a^1 + 3a^2 - a^3) + (2b^1 + 3b^2 - b^3)$$
$$= 0 + 0 = 0,$$

So we see that $a + b \in \mathcal{V}$

Compute

$$ca = (ca^1, ca^2, ca^3)$$

Then

$$2ca^1 + 3ca^2 - ca^3 = c(2a^1 + 3a^2 - a^3) = 0,$$

so $ca \in \mathcal{V}$.

Thus V is a subspace of E^3 that is a plane, so it is 2-dimensional.

Grad students

§1.2 #10 Let $\{x_1,\dots,x_n\}$ be a basis for E^n.

Let $v_1 = \frac{x_1}{\|x_1\|}$,

$$y_2 = x_2 - (x_2\cdot v_1)v_1$$

$$v_2 = \frac{y_2}{\|y_2\|}$$

$$y_3 = x_3 - (x_3\cdot v_2)v_2$$

$$v_3 = \frac{y_3}{\|y_3\|}$$

$$\vdots$$

$$y_n = x_n - (x_n\cdot v_{n-1})v_{n-1}$$

$$v_n = \frac{y_n}{\|y_n\|}$$

Show $\{v_1,\dots,v_n\}$ is an orthonormal basis for E^n.

Soln: Each $\|v_j\| = \left\|\frac{y_j}{\|y_j\|}\right\| = \frac{1}{\|y_j\|}\|y_j\| = 1$

Must verify $\{v_1,\dots,v_n\}$ is a basis.

so assume $\{x_1,x_2\}$ is a basis of E_2

Proof (just case $n=2$)

I won't ask the general proof on exam!!

Consider $n=2$ and equation $c_1v_1 + c_2v_2 = 0$

Need to argue $c_1=c_2=0$ is only soln to get that $\{v_1,v_2\}$ is indep.

By def of v_1 and v_2, we get

$$c_1\frac{x_1}{\|x_1\|} + c_2\frac{y_2}{\|y_2\|} = 0$$

By def of y_2, we get

$$\frac{c_1}{\|x_1\|}x_1 + \frac{c_2}{\|y_2\|}\left(x_2 - (x_2\cdot v_1)v_1\right) = 0$$

and again by def of v_1,

$$\frac{c_1}{\|x_1\|}x_1 + \frac{c_2}{\|y_2\|}x_2 - \frac{c_2}{\|y_2\|}\left(x_2\cdot\frac{x_1}{\|x_1\|}\right)\frac{x_1}{\|x_1\|} = 0$$

all scalars!

Simplify to

$$\underbrace{\left(\frac{c_1}{\|x_1\|} - \frac{c_2}{\|y_2\|}\left(x_2\cdot\frac{x_1}{\|x_1\|}\right)\right)}_{\text{call this scaler } \tilde{c}_1}x_1 + \underbrace{\frac{c_2}{\|y_2\|}}_{\text{call this scaler } \tilde{c}_2}x_2 = 0$$

So we have

$$\tilde{c}_1 x_1 + \tilde{c}_2 x_2 = 0$$

whose only soln is $\tilde{c}_1 = \tilde{c}_2 = 0$ since $\{x_1, x_2\}$ is a basis of E^2

Thus $\{v_1, v_2\}$ is independent. ("small enough to be indep")

Now we need to argue span $\{v_1, v_2\} = E^2$,

$$\text{Span}\{v_1, v_2\} := \{ c_1 v_1 + c_2 v_2 : c_1, c_2 \in \mathbb{R} \} = E^2$$

Clearly span $\{v_1, v_2\} \subseteq E^2$. Let $w = (w^1, w^2) \in E^2$. Need to find c_1, c_2 so that $c_1 v_1 + c_2 v_2 = w$.

By earlier, we know the left-hand side can be written as

$$\tilde{c}_1 x_1 + \tilde{c}_2 x_2 = w$$

But since $\{x_1, x_2\}$ is a basis, we know such $\tilde{c}_1$ and $\tilde{c}_2$ (and hence c_1 and c_2) exist. Thus span $\{v_1, v_2\} = E^2$. ("big enough to span")

Therefore $\{v_1, v_2\}$ is a basis of E^2.

Now check:

$$\|v_1\| = \left\| \frac{x_1}{\|x_1\|} \right\| = \frac{1}{\|x_1\|} \|x_1\| = 1$$

$$\|v_2\| = \left\| \frac{y_2}{\|y_2\|} \right\| = 1$$

and

$$v_1 \cdot v_2 = \left(\frac{x_1}{\|x_1\|} \right) \cdot \left(\frac{y}{\|y_2\|} \right)$$

def of y_1

$$= \frac{1}{\|x_1\| \, \|y_1\|} \left[x_1 \cdot \left(x_2 - (x_2 \cdot v_1) v_1 \right) \right]$$

$$= \frac{1}{\|x_1\| \, \|y_1\|} \left[x_1 \cdot x_2 - \left(x_2 \cdot \frac{x_1}{\|x_1\|} \right) x_1 \cdot \left(\frac{x_1}{\|x_1\|} \right) \right]$$

$$= \frac{1}{\|x_1\| \, \|y_1\|} \left[x_1 \cdot x_2 - \frac{1}{\cancel{\|x_1\|^2}} (x_2 \cdot x_1) \underbrace{(x_1 \cdot x_1)}_{= \cancel{\|x_1\|^2}} \right]$$

$$= 0,$$

as was to be shown.

§1.3 #6 Show $\{x : \|x - x_1\| = \|x - x_2\|\}$ for given $x_1, x_2 \in E^n$ is a hyperplane.

$\{x ; x \cdot z = c\}$

Proof: First note that

$$\|x - x_1\|^2 = (x - x_1) \cdot (x - x_1) = x \cdot x - 2x \cdot x_1 + x_1 \cdot x_1, \text{ and}$$

$$\|x - x_2\|^2 = (x - x_2) \cdot (x - x_2) = x \cdot x - 2x \cdot x_2 + x_2 \cdot x_2$$

So $\|x-x_1\| = \|x-x_2\|$ implies

$$\sqrt{x\cdot x - 2x\cdot x_1 + x_1\cdot x_1} = \sqrt{x\cdot x - 2x\cdot x_2 + x_2\cdot x_2}$$

$$\Downarrow$$

$$\cancel{x\cdot x} - 2x\cdot x_1 + x_1\cdot x_1 = \cancel{x\cdot x} - 2x\cdot x_2 + x_2\cdot x_2$$

$$x\cdot\underbrace{[2x_1 + 2x_2]}_{\text{call this } z} = \underbrace{x_2\cdot x_2 - x_1\cdot x_1}_{\text{call this } c}$$

So we get

$$\{x : x\cdot z = c\}$$

which is a hyperplane.

#7 Show that $\{x : \|x-x_1\| = c\|x-x_2\|\}$ where $x_1, x_2 \in E^n$ and $0<c<1$ is an (n−1)-sphere.

looks like $\{y : \underbrace{\|x-y\| < \delta}\}$

$\|x-y\|^2 = (x-y)\cdot(x-y)$
$= x\cdot x - 2x\cdot y + y\cdot y$

$$\Downarrow$$

$$\sqrt{x\cdot x - 2x\cdot y + y\cdot y} < \delta$$

Soln: Again we get

$\|x-x_1\|^2 = (x-x_1)\cdot(x-x_1) = x\cdot x - 2x\cdot x_1 + x_1\cdot x_1$, and

$\|x-x_2\|^2 = (x-x_2)\cdot(x-x_2) = x\cdot x - 2x\cdot x_2 + x_2\cdot x_2$

So we obtain

$$\|x-x_1\| = c\|x-x_2\|$$

$$\downarrow$$

$$\sqrt{x\cdot x - 2x\cdot x_1 + x_1\cdot x_1} = c\sqrt{x\cdot x - 2x\cdot x_2 + x_2\cdot x_2}$$

$$\downarrow \text{square both sides}$$

$$x\cdot x - 2x\cdot x_1 + x_1\cdot x_1 = c^2\left(\underbrace{x\cdot x - 2x\cdot x_2}_{} + x_2\cdot x_2\right)$$

$$(1-c^2)x\cdot x - 2x\cdot(x_1 - c^2x_2) + x_1\cdot x_1 = c^2 x_2\cdot x_2$$

$$x\cdot x - 2x\cdot\underbrace{\frac{(x_1-c^2x_2)}{1-c^2}}_{y} + \frac{x_1\cdot x_1}{1-c^2} = \frac{c^2}{1-c^2}\, x_2\cdot x_2$$

$$y\cdot y = \left(\frac{1}{1-c^2}\right)^2\left[x_1\cdot x_1 - 2c^2x_1\cdot x_2 + c^4x_2\cdot x_2\right]$$

$$\Rightarrow \frac{x_1\cdot x_1}{1-c^2} = (1-c^2)\,y\cdot y - \frac{2c^2}{(1-c^2)^2}x_1\cdot x_2 + \frac{c^4}{(1-c^2)^2}x_2\cdot x_2$$

So we have

keep on left

$$x\cdot x - 2x\cdot y + \left[(1-c^2)y\cdot y - \frac{2c^2}{(1-c^2)^2}x_1\cdot x_2 + \frac{c^4}{(1-c^2)^2}x_2\cdot x_2\right] = \frac{c^2}{1-c^2}x_2\cdot x_2$$

$$x\cdot x - 2x\cdot y + y\cdot y = \underbrace{\frac{c^2}{1-c^2}x_2\cdot x_1 + c^2 y\cdot y + \frac{2c^2}{(1-c^2)^2}x_1\cdot x_2}_{\text{call this } \delta} - \frac{c^4}{(1-c^2)^2}x_2\cdot x_2 < \delta$$

So we have

$$\|x-y\| = x\cdot x - 2x\cdot y + y\cdot y < \delta$$

which is the defining property of the $(n-1)$-sphere.

#12 Let P be the hyperplane $\{x : z\cdot x = c\}$

Prove line thru any 2 pts of P is contained in P.

Why's this mean P is convex?

Soln: Let $x, y \in P$, so $z\cdot x = c$ and $z\cdot y = c$.

The line thru x and y is all pts of form

$$tx + (1-t)y, \quad t\in\mathbb{R}$$

But compute

$$z\cdot\big(tx + (1-t)y\big) = t(z\cdot x) + (1-t)(z\cdot y)$$
$$= ct + c(1-t)$$
$$= c$$

Thus all points on the line lie in the plane.

This shows P convex because the line between any two points of P is in P, so in particular the line segment b/w two such points is in P, which is the def. of convex.