A bounded set A is *Jordan measurable* if its characteristic function 1_A is Riemann integrable. It can be shown that A is Jordan measurable if and only if fr A is a null set [1, p.256].

If $A = [a, b]$, a closed interval of E^1, then the definition of Riemann integral given above can rather easily be shown to agree with Riemann's original definition of integral as the limit of sums.

PROBLEMS

1. Let $f(x) = 2x - x^2$ if $0 \le x \le 2$, $f(x) = 0$ otherwise. Using the notation in the proof of Lemma 1, describe the sets $A_1, \ldots, A_m$. Sketch the step functions ϕ and ψ in the case $m = 4$.

2. In each case show that f is integrable over A.
   (a) $f(x) = x^2 \exp x$, $A = [0, a]$.
   (b) $f(x) = \sin(1/x)$ if $x \neq 0$, $f(0) = 5$, $A = [-1, 1]$.
   (c) $f(x, y) = (x^4 - y^2)/(x^2 - y)$, $A = \{(x, y): |x| \le 1, |y| \le 1, x^2 \neq y\}$.
   (d) $f(x) = 0$ if x is irrational, $f(x) = 1/q$ if $x = p/q$ where p and q are integers with no common factor, $A = (0, 1]$.
   (e) $f(x) = 1$ if x is irrational, $f(x) = 0$ if x is rational, $A = [a, b]$.

3. For each part of Problem 2 describe the sets $\{\mathbf{x} \in A : f(\mathbf{x}) > c\}$.

4. Show that if f, g, and h are integrable over A and $|f(\mathbf{x}) - g(\mathbf{x})| \le h(\mathbf{x})$ for every $\mathbf{x} \in A$, then $|\int_A f\,dV - \int_A g\,dV| \le \int_A h\,dV$.

5. Let f be of class $C^{(2)}$ on $[0, a]$ and $b = \max\{|f''(x)| : 0 \le x \le a\}$. Let $g(x) = f(0) + f'(0)x$. Using Problem 4, show that $|\int_0^a f\,dx - af(0) - a^2 f'(0)/2| \le a^3 b/6$. Use this result to estimate $\int_0^{1/2} \exp(-x^2/2)dx$.

6. (*Mean value theorem for integrals.*) Let A be compact and connected. Let f be continuous on A and g be integrable over A with $g(\mathbf{x}) \ge 0$ for every $\mathbf{x} \in A$. Prove that there exists $\mathbf{x}^* \in A$ such that
$$\int_A fg\,dV = f(\mathbf{x}^*) \int_A g\,dV.$$
   [*Hint*: Let C and c be the maximum and minimum values of f on A. Then $cg \le fg \le Cg$. Use (2) and (4) of Theorem 5.4 and the intermediate value theorem.]

5.5 Iterated integrals

Thus far we have given no effective procedure for the actual evaluation of integrals. One method for doing this is by writing the integral as an iterated integral and applying the fundamental theorem of calculus. Let $1 \le s < n$. In most cases we shall take $s = 1$ or $s = n - 1$. Then E^n can be regarded as the cartesian product $E^s \times E^{n-s}$. Let us write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where
$$\mathbf{x}' = (x^1, \ldots, x^s) \in E^s, \qquad \mathbf{x}'' = (x^{s+1}, \ldots, x^n) \in E^{n-s}.$$

Let A be a set and f a function whose domain contains A. In the present section we assume that A is compact, and that f is continuous on A. These

PROBLEMS

1. Find the area and also the centroid of:
   (a) $\{(x, y): x^2 \le y \le x + 2\}$. (b) $\{(x, y): |y| - 1 \le x \le \sqrt{1 - y^2}\}$.

2. Express the iterated integral

$$\int_0^1 dy \int_0^{f(y)} xy\, dx, \text{ where } f(y) = \min[1, \log(1/y)],$$

   as an integral over a set $A \subset E^2$, and then as an iterated integral in the opposite order. Evaluate it.

3. Express as an iterated triple integral:

$$\int_A f\, dV_3, \text{ where } A = \{(x, y, z): x^2 + z^2 \le y^2 \le 8 - (x^2 + z^2)\}.$$

4. Find the volume of

$$\{(x, y, z): |x| + |y| + |z| \le 2, |x| \le 1, |y| \le 1\}.$$

5. Find the volume of

$$\{(x, y, z): |x| + |y| + |z| \le 2, z^2 \le y\}.$$

6. (a) Suppose that $f(x, y) = g(x)h(y)$ for every $(x, y) \in A$ and that $A = R \times S$. Show that

$$\int_A f\, dV_2 = \left(\int_R g\, dx\right)\left(\int_S h\, dy\right).$$

   (b) Evaluate $\int_0^1 dx \int_0^1 \exp(x + y)dy$.
   (c) Evaluate $\int_0^\pi dy \int_0^{\pi/2} xy \cos(x + y)dx$.

7. Let α_n be the measure of the unit n-ball $\{\mathbf{x}: |\mathbf{x}| \le 1\}$. Show that

$$\alpha_n = 2\alpha_{n-1} \int_0^1 (1 - u^2)^{(n-1)/2} du.$$

   Show that $\alpha_4 = \pi^2/2$. (In Section 5.9 we give a general formula for α_n.)

8. Write points of E^{n+1} as $(\mathbf{x}, z)$, where $\mathbf{x} = (x^1, \dots, x^n)$. Let

$$A = \{(\mathbf{x}, z): 0 \le z \le 1 - |\mathbf{x}|^2\}.$$

   Show that $V_{n+1}(A) = 2\alpha(n)/(n + 2)$.

9. Let Σ be the standard n-simplex.
   (a) Show that the centroid of Σ is at the barycenter.
   (b) Show that the second moment of Σ about the $(n - r)$-dimensional plane spanned by $\mathbf{e}_{r+1}, \dots, \mathbf{e}_n$ is $2r/(n + 2)!$

10. Recall the definition of uniform convergence of a sequence of functions (Section 2.10). Show that the sequence $F_1, F_2, \dots$ constructed in the proof of Theorem 5.6 converges uniformly to f_A if and only if $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \text{fr } A$. [*Hint*: f is uniformly continuous on the compact set A (see Problem 8, Section 2.5).]