and lower Riemann integrals by $\bar{S}(f)$ and $\underline{S}(f)$. Then

(5.19) $$\underline{S}(f) \le \underline{\int} f\, dV \le \overline{\int} f\, dV \le \bar{S}(f).$$

If $\underline{S}(f) = \bar{S}(f)$, then f is called *Riemann integrable*. Their common value $S(f)$ is the *Riemann integral* of f. From (5.19), if f is Riemann integrable, then f is integrable [in the sense of (5.16)] and

(5.20) $$S(f) = \int f\, dV.$$

It can be proved that a bounded function f with compact support is Riemann integrable if and only if $V(\{\mathbf{x} : f \text{ is discontinuous at } \mathbf{x}\}) = 0$ [1, pp. 230 and 260].

Problems

1. Determine whether f is bounded. Find its support.
   (a) $f(x) = x - |x|$.
   (b) $f(x, y) = x \exp(-x^2 - y^2)$.
   (c) $f(x, y) = 1$ if either x or y is a rational number, $f(x, y) = 0$ if both x and y are irrational.
   (d) $f(x, y) = (x - y)|x + y| - (x + y)|x - y|$ if $|x| + |y| < 1$, $f(x, y) = 0$ if $|x| + |y| \ge 1$. Illustrate with a sketch.
2. Let $[a]$ denote the largest integer which is no greater than a (for instance, $[\pi] = 3$). Let $\phi(x, y) = [x + y]$ if $0 \le x < r, 0 \le y < s$, where r and s are positive integers. For all other (x, y) let $\phi(x, y) = 0$. Show that
$$\int \phi\, dV_2 = \frac{rs(r + s - 1)}{2}.$$
3. Let a unit square be divided into a square of side $(4m + 1)^{-1}$ in the center and $2m$ annular figures of equal width $(4m + 1)^{-1}$ surrounding it, as shown in Figure 5.5.

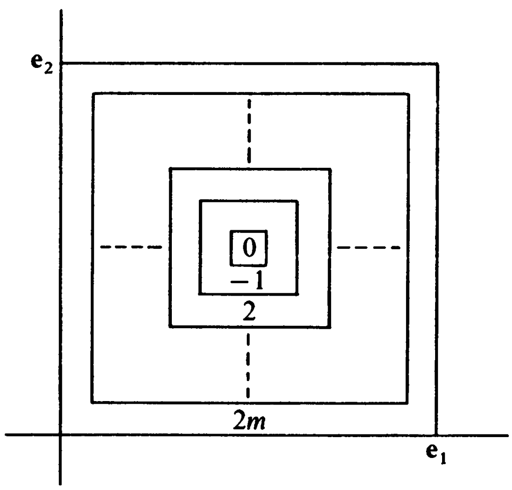

Figure 5.5

Let $\phi(x, y) = 0$ for (x, y) in the small square or outside the large square. Let $\phi(x, y) = (-1)^k k$ in the kth annular figure, $k = 1, \dots, 2m$. Show that

$$\int \phi \, dV_2 = \frac{8m(2m+1)}{(4m+1)^2}.$$

What is this approximately when m is large?

4. (a) Show that if f is integrable, then $\int (cf) dV = c \int f \, dV$. [*Hint*: Show that this is true if $c \geq 0$, and that $-\underline{\int} g \, dV = \overline{\int} (-g) dV$ for every g. If $c < 0$, set $g = cf$ and $g = -cf$.]
   (b) Show that $\overline{\int} f \, dV \leq \overline{\int} g \, dV$ if $f \leq g$.

5.4 Integrals over bounded sets

Let A be a bounded measurable set and f be a function that is bounded on A. More precisely, the domain of f contains A and there is a number C such that $|f(\mathbf{x})| \leq C$ for every $\mathbf{x} \in A$. Let us consider a new function with the same values as f on A and the value 0 otherwise. This function is denoted by f_A. Thus

$$f_A(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \notin A. \end{cases}$$

The function f_A is bounded and has compact support. The values of f outside A should contribute nothing to the integral of f over A.

Definition. The function f is *integrable over* A if f_A is an integrable function. The *integral of* f *over* A is the number

(5.21) $$\int_A f \, dV = \int f_A \, dV.$$

In later sections it is sometimes convenient to use the notation $\int_A f(\mathbf{x}) dV(\mathbf{x})$ for the integral $\int_A f \, dV$. Moreover, we sometimes emphasize the role of the dimension n by writing dV_n instead of dV. When $n = 1$, we usually write $\int_A f(x) dx$ instead of $\int_A f(x) dV_1(x)$.

Proposition 5.4 implies that sums and scalar multiples of functions integrable over A are also integrable over A. Theorem 5.5 gives a widely applicable condition for integrability of f. In the meantime, we summarize a number of properties of the integral in the following theorem.

Theorem 5.4. *If all the integrals involved exist, then:*

(1) $\int_A (f + g) dV = \int_A f \, dV + \int_A g \, dV$.
(2) $\int_A (cf) dV = c \int_A f \, dV$.
(3) $\int_A 1 \, dV = V(A)$.
(4) *If* $f(\mathbf{x}) \leq g(\mathbf{x})$ *for every* $\mathbf{x} \in A$, *then* $\int_A f \, dV \leq \int_A g \, dV$.
(5) *If* $|f(\mathbf{x})| \leq C$ *for every* $\mathbf{x} \in A$, *then* $|\int_A f \, dV| \leq \int_A |f| \, dV \leq CV(A)$.
(6) *If* A *is a null set, then* $\int_A f \, dV = 0$.
(7) *If* $A \cap B$ *is a null set, then* $\int_{A \cup B} f \, dV = \int_A f \, dV + \int_B f \, dV$.