Hence A is a null set. Since $V_1(A) + V_1[(0, 1) - A] = V_1[(0, 1)] = 1$, the set of irrational numbers in (0, 1) has measure 1 and therefore must be an uncountable set.

EXAMPLE 5. Let $A \subset M$, where M is an $(n - 1)$-manifold. It is plausible that the n-dimensional measure of A is 0, and this fact is proved in Section 5.8. Hence any such set A is a null set.

A sequence of sets $A_1, A_2, \ldots$ is called *monotone* if either $A_1 \subset A_2 \subset \cdots$ or $A_1 \supset A_2 \supset \cdots$. In the first instance the sequence is called *nondecreasing*, and in the second instance *nonincreasing*.

Theorem 5.3

(a) *Let $A_1, A_2, \ldots$ be a nondecreasing sequence of measurable sets. Then*

$$V\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) = \lim_{\nu\to\infty} V(A_\nu) \tag{5.11}$$

(b) *Let $A_1, A_2, \ldots$ be a nonincreasing sequence of measurable sets, such that $V(A_1) < \infty$. Then*

$$V\left(\bigcap_{\nu=1}^{\infty} A_\nu\right) = \lim_{\nu\to\infty} V(A_\nu) \tag{5.12}$$

PROOF. Let us prove the theorem under the assumption that there is a spherical ball U such that $A_\nu \subset U$ for each $\nu = 1, 2, \ldots$. This restriction will be removed in Section 5.10. To prove (a), define $B_1, B_2, \ldots$ as in the proofs of Theorems 5.1 and 5.2. Then

$$V\left(\bigcup_{\nu=1}^{\infty} A_\nu\right) = \sum_{k=1}^{\infty} V(B_k).$$

Since $A_1 \subset A_2 \subset \cdots$, we have $B_1 \cup \cdots \cup B_\nu = A_\nu$. Therefore

$$V(A_\nu) = \sum_{k=1}^{\nu} V(B_k).$$

We get (5.11) by taking the limit as $\nu \to \infty$. To get (5.12), we apply (5.11) to the nondecreasing sequence of sets $C_\nu = U - A_\nu$, and note that $V(C_\nu) = V(U) - V(A_\nu)$ since $A_\nu \subset U$. □

EXAMPLE 6. To see the need for the assumption $V(A_1) < \infty$ in Theorem 5.3(b), let $n = 1$ and $A_\nu = [\nu, \infty)$. Then $A_1 \supset A_2 \supset \cdots$ and $V(A_\nu) = +\infty$ for each $\nu = 1, 2, \ldots$. However, $A_1 \cap A_2 \cap \cdots$ is empty, and hence $V(A_1 \cap A_2 \cap \cdots) = 0$.

PROBLEMS

In 1, 2, and 3 assume that the sets are bounded.

1. Let A and B be measurable. Show that·
(a) $V(A - B) = V(A) - V(A \cap B)$.
(b) $V(A \cup B) + V(A \cap B) = V(A) + V(B)$.

2. Show that if A, B, and C are measurable, then

$$V(A \cup B \cup C) = V(A) + V(B) + V(C) - V(A \cap B) - V(A \cap C) \\ - V(B \cap C) + V(A \cap B \cap C).$$

3. Show that if A is measurable and B is a null set, then

$$V(A \cup B) = V(A - B) = V(A).$$

4. Let $A = A_1 \cup A_2 \cup \cdots$, where $A_k = \{(x, y): x = 1/k, 0 \le y \le 1\}$ for $k = 1, 2, \ldots$ Show that $V_2(A) = 0$.

5. Let A_0 be the circular disk with center (0, 0) and radius 1. For $k = 1, 2, \ldots$, let A_k be the circular disk with center $(1 - 4^{-k})\mathbf{e}_1$ and radius 4^{-k-1}. Let $A = A_0 - (A_1 \cup A_2 \cup \cdots)$. Find $V_2(A)$.

6. Prove Lemma 1. [*Hint*: Consider the collection of all intervals I such that $I \subset G$. The interiors int I of these intervals form an open covering of K.]

7. Prove Corollary 1 to Proposition 5.1a. [*Hint*: If $K \subset G$, then $G = K \cup (G - K)$. By Proposition 5.1a, $V(G) = V(K) + V(G - K)$.]

8. (a) Show that if A and B are countable sets, then $A \cup B$ is countable.
(b) Show that if $B \subset A$ and A is countable, then B is countable.
(c) Show that if $A_1, A_2, \ldots$ are countable sets, then $A_1 \cup A_2 \cup \cdots$ is countable.

9. Let $A = \{x_1, x_2, \ldots\}$ be a countable subset of (0, 1). Given $0 < \varepsilon < 1$, let $\varepsilon_k = \varepsilon 2^{-k-1}$, $I_k = (x_k - \varepsilon_k, x_k + \varepsilon_k)$, and $G = I_1 \cup I_2 \cup \cdots$.
(a) Show that $V_1(G) \le \varepsilon$.
(b) In particular, let A be the set of rational numbers in (0, 1). Let $K = [0, 1] - G$. Then K is a compact subset of the irrational numbers. Show that $V_1(K) \ge 1 - \varepsilon$.
(c) Show that $K = \text{fr } K$.

10. Let C be the Cantor set, defined in Problem 5, Section 2.4.
(a) Show that C is a null set ($V_1(C) = 0$).
(b) Show that $x \in C$ if and only if $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i = 0$ or 2, $i = 1, 2, \ldots$.
(c) Let $f(x) = \sum_{i=1}^{\infty} a_i 2^{-i-1}$ for $x \in C$. Show that $f(C) = [0, 1]$. Hence C is uncountable.
(d) For x in the kth interval of A_j (Problem 5, Section 2.4) let $f(x)$ have the constant value $(2k - 1)2^{-j}$, $k = 1, 2, \ldots, 2^{j-1}$, $j = 1, 2, \ldots$. Show that f is continuous and nondecreasing on [0, 1]. [*Note*: f is called the *Cantor function*.]

11. Show that any straight line in E^2 has area 0.