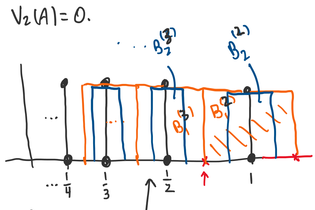
HW10 MTH 428/528 Spring 2025

Monday, April 21, 2025 8:14 AM

§5.2 #4 Let $A = A_1 \cup A_2 \cup \dots$, $A_k = \{(x,y) : x = \frac{1}{k}, 0 \le y \le 1\}$ for $k = 1, 2, \dots$

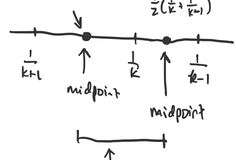
Show $V_2(A) = 0$.

Soln:



$\frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right) = \frac{2k+1}{2k(k+1)}$

$\frac{1}{2}\left(\frac{2k-1}{k(k-1)}\right) = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-1}\right)$



distance b/w midpoints

$= \frac{1}{2}\left(\frac{2k-1}{k(k-1)} - \frac{2k+1}{(k)(k+1)}\right)$

$= \frac{1}{2}\left[\frac{2k^2 - k + 2k - 1 - 2k^2 - 2k - k - 1}{k(k-1)(k+1)}\right]$

$= \frac{1}{(k-1)(k+1)}$

width of $B_1^{(k)}$

$\Downarrow$

width of $B_m^{(k)}$ is

$\frac{1}{2^m}\left[\frac{1}{(k-1)(k+1)}\right]$

Let $B_1^{(k)}$ be as shown, each rectangle has width $\frac{1}{k}$

Let $B_2^{(k)}$ be same but half as wide, as pictured

And in general let $B_{n+1}^{(k)}$ be $B_n^{(k)}$ with half the area.

$$V\left(\bigcup_{k=1}^{\infty} B_m^{(k)}\right) = \sum_{k=2}^{\infty} V\left(B_m^{(k)}\right) = \underbrace{\frac{1}{2^m}\sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)}}_{\text{this converges! Call it } W} = \frac{W}{2^m}$$

Now as $m \to \infty$, we have always

$B_{m+1}^{(k)} \subseteq B_m^{(k)}$ and we have

$$A = \bigcap_{m=1}^{\infty} \bigcup_{k=2}^{\infty} B_m^{(k)}$$

So A can be written as sequence of nonincr msbl sets

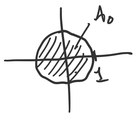
Thm 5.3(b) $\Longrightarrow$ $$V(A) = \lim_{m\to\infty} V\left(\bigcup_{k=1}^{\infty} B_m^{(k)}\right) = \lim_{m\to\infty} \frac{W}{2^m} = 0$$

$1 - 4^{-k}$

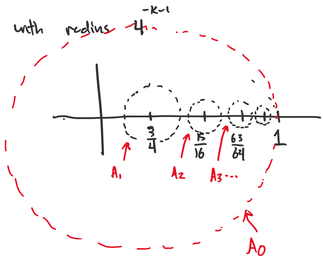
$= 1 - \frac{1}{4^k}$

$= \frac{4^k - 1}{4^k}$

#5



Let A_k be circular disk w/ center $\underbrace{(1-4^{-k})}e_1$ with radius 4^{-k-1}



$k =$	$1-4^{-k} =$	$4^{-k-1} =$
1	3/4	1/16
2	15/16	1/64
3	63/64	1/256
4	255/256	1/1024
⋮		
k	$\frac{4^k - 1}{4^k}$	

Let $A = A_0 \setminus (A_1 \cup A_2 \cup \dots)$

Find $V_2(A)$.

Soln: Each circle A_k, $k = 1, 2, 3, \dots$ has radius 4^{-k-1}, so

$$V(A_k) = \pi\left(4^{-k-1}\right)^2 = \pi 4^{-2k-2}$$

So area of

$$V\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} V(A_k) = \pi \sum_{k=1}^{\infty} 4^{-2k-2}$$

$A_{k+1} \cap A_k = \phi$!

$4^{-2k} = \left(\frac{1}{4^2}\right)^k$

$$= \frac{\pi}{16}\sum_{k=1}^{\infty}\left(\frac{1}{16}\right)^k = \frac{\pi}{16}\,\frac{\frac{1}{16}}{1 - \frac{1}{16}}$$

geo. series

$$= \frac{\pi}{16}\left(\frac{1}{16-1}\right)$$

$$= \frac{\pi}{15 \cdot 16}$$

Now $V(A_0) = \pi(1^2) = \pi$

So,

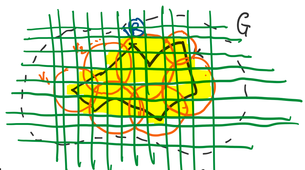
$$V(A) = V(A_0) - V(A_1 \cup A_2 \cup \dots)$$

$$= \pi - \frac{\pi}{15 \cdot 16}$$

Grad students

#6 Prove lemma 1: If K cpt, G open with $K \subset G$, then $\exists$ figure Y s.t. $K \subset \text{int}(Y)$ and $Y \subset G$.

Proof:



For each $w \in K$ let $\mathcal{U}_w$ be an open ball s.t. $w \in \mathcal{U}_w$ and $\mathcal{U}_w \subset G$, which we can do because G is open. Then $\mathcal{C} = \{\mathcal{U}_w : w \in K\}$ is an open cover of K contained in G.

Since K cpt, there is a finite open cover $\mathcal{F} \subseteq \mathcal{C}$, $\mathcal{F} = \{V_1, \dots, V_n\}$.

Let R be distance from $cl(V_1 \cup V_2 \dots V_n)$ and the boundary of G.

Choose grid points spaced by $\frac{R}{2}$. Then a figure Y can be picked from these points satisfying the requirements!