Quiz 2 MTH 416/616 Spring 2025

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \xrightarrow{\text{eigenvalues}} \lambda_1 = i, \lambda_2 = -i$$

$$M_0 = I \quad, \quad M_1 = A - iI = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$$

P solves

$$P' = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix} P, \quad P(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} P_1' = i P, \quad P(0) = 1 \longrightarrow P(t) = e^{it} \\ P_2' = P_1 - P_2 i, \quad P(0) = 0 \end{cases}$$

apply variation of parameters to

$$P_2' = e^{it} - P_2 i, \quad P_2(0) = 0 \qquad \boxed{-i^2 = 1}$$

$$P_2(t) = -\frac{i}{2}\left[e^{it} - e^{-it}\right] = -\frac{i}{2}(2i)\sin(t) = \sin(t)$$

Therefore,

$$e^{At} = P_1 M_0 + P_2 M_1$$

$$= e^{it} I + \sin(t)\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$$

$$= \begin{bmatrix} e^{it} + \sin(t)(1-i) & \sin(t) \\ -2\sin(t) & e^{it} - (1+i)\sin(t) \end{bmatrix}$$

$$\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) \\ \quad = \cos(\theta) - i\sin(\theta) \\ \qquad \downarrow \\ e^{i\theta} - e^{-i\theta} = 2i\sin\theta \end{array}}$$

$$= \begin{bmatrix} \cos(t) + i\sin(t) + \sin(t)(1-i) & \sin(t) \\ -2\sin(t) & \cos(t) + i\sin(t) - (1+i)\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$