HW5 MTH 416/616 Spring 2025

Friday, March 7, 2025 8:02 AM

Ch.2 #2.27 Find e^{At} where

(i) $A=\begin{bmatrix}2&1\\-9&-4\end{bmatrix}$

Soln: evals

$$\det\left(\begin{bmatrix}2-\lambda&1\\-9&-4-\lambda\end{bmatrix}\right)=(2-\lambda)(-4-\lambda)+9\overset{set}{=}0$$

$$-8-2\lambda+4\lambda+\lambda^2+9=0$$

$$\lambda^2+2\lambda+1=0\quad(\lambda+1)^2=0$$

$\lambda=-1$, repeated root

$$M_0=I\qquad M_1=A-\lambda I=\begin{bmatrix}2-(-1)&1\\-9&-4-(-1)\end{bmatrix}=\begin{bmatrix}3&1\\-9&-3\end{bmatrix}$$

p solves

$$p'=\begin{bmatrix}-1&0\\1&-1\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{cases}p_1'=-p_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{-t}\\ p_2'=p_1-p_2,\ p_2(0)=0\longrightarrow p_2'=p_2+e^{-t},\ p_2(0)=1\end{cases}$$

var of constant

$x'=p(t)x+q(t)$, $x(t_0)=x_0$

$$\Downarrow$$

$$x(t)=e^{\int_{t_0}^t p(\tau)d\tau}x_0+e^{\int_{t_0}^t p(\tau)d\tau}\int_{t_0}^t e^{-\int_{t_0}^s p(\tau)d\tau}q(s)ds$$

$p(t)=-1$ $\quad q(t)=e^{-t}$

$$e^{\int_0^t p(\tau)d\tau}=e^{-t}\qquad e^{-\int_0^s p(\tau)d\tau}=e^{s}$$

$$p_2(t)=0+e^{-t}\int_0^t e^{s}e^{-s}ds=te^{-t}$$

By Putzer thm,

$$e^{At}=p_1M_0+p_2M_1$$

$$=e^{-t}\begin{bmatrix}1&0\\0&1\end{bmatrix}+te^{-t}\begin{bmatrix}3&1\\-9&-3\end{bmatrix}$$

$$=\begin{bmatrix}e^{-t}+3te^{-t}&te^{-t}\\-9te^{-t}&e^{-t}-3te^{-t}\end{bmatrix}$$

(ii) $A=\begin{bmatrix}10&4\\-9&-2\end{bmatrix}$

$\hookrightarrow$ eigenvalues $\lambda=4$, repeated

$$M_0=I\qquad M_1=A-4I=\begin{bmatrix}6&4\\-9&-6\end{bmatrix}$$

p solves

$$p'=\begin{bmatrix}4&0\\1&4\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{cases}p_1'=4p_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{4t}\\ p_2'=p_1+4p_2,\ p_2(0)=0\xrightarrow{\text{var of const}} p_2(t)=te^{4t}\end{cases}$$

$$e^{At}=M_0p_1+M_1p_2=e^{4t}\begin{bmatrix}1&0\\0&1\end{bmatrix}+te^{4t}\begin{bmatrix}6&4\\-9&-6\end{bmatrix}$$

$$=\begin{bmatrix}e^{4t}+6te^{4t}&4te^{4t}\\-9te^{4t}&e^{4t}-6te^{4t}\end{bmatrix}$$

(iii) $A=\begin{bmatrix}2&-2&2\\0&1&1\\-4&8&3\end{bmatrix}$

$\hookrightarrow$ evals: 3, 2, 1

$$M_0=I,\quad M_1=A-3I=\begin{bmatrix}-1&-2&2\\0&-2&1\\-4&8&0\end{bmatrix},\quad M_2=A-2I=\begin{bmatrix}0&-2&2\\0&-1&1\\-4&8&1\end{bmatrix}$$

p solves

$$p'=\begin{bmatrix}3&0&0\\1&2&0\\0&1&1\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{cases}p_1'=3p_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{3t}\\ p_2'=p_1+2p_2,\ p_2(0)=0\xrightarrow{\text{var of const}} p_2(t)=e^{3t}-e^{2t}\\ p_3'=p_2+p_3,\ p_3(0)=0\longrightarrow p_3(t)=\frac12e^t(e^{2t}-2e^t+1)=\frac12(e^{3t}-2e^{2t}+e^t)\end{cases}$$

$$\longrightarrow e^{At}=M_0p_1+M_1p_2+M_2p_3$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}e^{3t}+\begin{bmatrix}-1&-2&2\\0&-2&1\\-4&8&0\end{bmatrix}(e^{3t}-e^{2t})+\begin{bmatrix}0&-2&2\\0&-1&1\\-4&8&1\end{bmatrix}\frac12(e^{3t}-2e^{2t}+e^t)$$

2.28 Use Putzer algorithm to find e^{At}

(i) $A=\begin{bmatrix}1&2\\-5&-1\end{bmatrix}$

$\hookrightarrow$ evals: $\lambda_1=3i$, $\lambda_2=-3i$

Here $n=1$, so $e^{At}=\sum_{k=0}^{1}p_{k+1}(t)M_k$, where

$$M_0=I,\quad M_1=A-3iI=\begin{bmatrix}1-3i&2\\-5&-1-3i\end{bmatrix}$$

and p solves

$$p'=\underbrace{\begin{bmatrix}3i&0\\1&-3i\end{bmatrix}}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$e^{i\theta}=\cos\theta+i\sin\theta$

$e^{-i\theta}=\cos(-\theta)+i\sin(-\theta)$

$=\cos(\theta)-i\sin(\theta)$

$\Rightarrow e^{i\theta}-e^{-i\theta}=2i\sin(\theta)$

$\Rightarrow \frac{e^{i\theta}-e^{-i\theta}}{2}=i\sin(\theta)$

$$\rightarrow p_1'=3ip_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{3it}=\cos(3t)+i\sin(3t)$$

$$p_2'=p_1-3ip_2,\ p_2(0)=0\longrightarrow p_2'=e^{3it}-3ip_2$$

$\downarrow$ variation of parameters

$$p_2(t)=\frac{i}{6}\left[e^{-3it}-e^{3it}\right]=-\frac{i}{3}\left[\frac{e^{i(3t)}-e^{-i(3t)}}{2}\right]=-\frac{i}{3}(i\sin(3t))$$

$$=\frac13\sin(3t)$$

Thus,

$$e^{At}=p_1M_0+p_2M_1=(\cos(3t)+i\sin(3t))I+\frac13\sin(3t)\begin{bmatrix}1-3i&2\\-5&-1-3i\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1-\cancel{3i}}{3}\sin(3t)+\cos(3t)+\cancel{i\sin(3t)}&\frac23\sin(3t)\\-\frac53\sin(3t)&-\frac{(1+\cancel{3i})}{3}\sin(3t)+\cos(3t)+\cancel{i\sin(3t)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac13\sin(3t)+\cos(3t)&\frac23\sin(3t)\\-\frac53\sin(3t)&-\frac13\sin(3t)+\cos(3t)\end{bmatrix}$$

(ii) $A=\begin{bmatrix}2&3\\-3&2\end{bmatrix}\longrightarrow$ eigenpairs: $\lambda_1=2+3i$, $v_1=\begin{bmatrix}-i\\1\end{bmatrix}$; $\lambda_2=2-3i$, $v_2=\begin{bmatrix}i\\1\end{bmatrix}$

$$M_0=I\qquad M_1=A-(2+3i)\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-3i&3\\-3&-3i\end{bmatrix}$$

and p solves

$$p'=\begin{bmatrix}2+3i&0\\1&2-3i\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$p_1'=(2+3i)p_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{(2+3i)t}=e^{2t}[\cos(3t)+i\sin(3t)]$$

$$p_2'=p_1+(2-3i)p_2,\ p_2(0)=0\xrightarrow{\text{var of param}}p_2(t)=\frac{-ie^{2t}}{6}\left[e^{3it}-e^{-3it}\right]$$

$$=-\frac{i}{3}e^{2t}\underbrace{\left[\frac{e^{3it}-e^{-3it}}{2}\right]}_{=i\sin(3t)\text{ from }e^{i\theta}=\cos\theta+i\sin\theta}$$

$$=\frac{e^{2t}}{3}\sin(3t)$$

Thus,

$$e^{At}=p_1M_0+p_2M_1=e^{2t}[\cos(3t)+i\sin(3t)]\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac{e^{2t}}{3}\sin(3t)\begin{bmatrix}-3i&3\\-3&-3i\end{bmatrix}$$

$$=e^{2t}\begin{bmatrix}\cos(3t)+\cancel{i\sin(3t)}-\cancel{i\sin(3t)}&\sin(3t)\\-\sin(3t)&\cos(3t)+\cancel{i\sin(3t)}-\cancel{i\sin(3t)}\end{bmatrix}$$

$$=e^{2t}\begin{bmatrix}\cos(3t)&\sin(3t)\\-\sin(3t)&\cos(3t)\end{bmatrix}$$

(iii) $A=\begin{bmatrix}0&2\\-1&2\end{bmatrix}$

$\hookrightarrow$ evals: $\lambda_1=1+i$, $v_1=\begin{bmatrix}1-i\\1\end{bmatrix}$; $\lambda_2=1-i$, $v_2=\begin{bmatrix}1+i\\1\end{bmatrix}$

$$M_0=I,\quad M_1=A-(1+i)I=\begin{bmatrix}-1-i&2\\-1&1-i\end{bmatrix}\quad 2-(1+i)$$

p solves

$$p'=\begin{bmatrix}1+i&0\\1&1-i\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{cases}p_1'=(1+i)p_1,\ p_1(0)=1\\ p_2'=p_1+(1-i)p_2,\ p_2(0)=0\end{cases}\longrightarrow\begin{matrix}p_1(t)=e^{(1+i)t}=e^t[\cos(t)+i\sin(t)]\\ p_2(t)=e^t\sin(t)\end{matrix}$$

$$\Rightarrow e^{At}=p_1M_0+p_2M_1=e^t[\cos(t)+i\sin(t)]\begin{bmatrix}1&0\\0&1\end{bmatrix}+te^t\sin(t)\begin{bmatrix}-1-i&2\\-1&1-i\end{bmatrix}$$

$$=e^t\begin{bmatrix}\cos(t)-\sin(t)&2\sin(t)\\-\sin(t)&\cos(t)+\sin(t)\end{bmatrix}$$

2.29 Use Putzer to solve...

(i) $x'=\overset{A}{\overbrace{\begin{bmatrix}1&1\\1&1\end{bmatrix}}}x$ $\quad$ evals $\lambda_1=2$, $\lambda_2=0$

Find e^{At}: $M_0=I$, $M_1=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}$

$$p'=\begin{bmatrix}2&0\\1&0\end{bmatrix}p,\quad p(0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$p_1'=2p_1,\ p_1(0)=1\longrightarrow p_1(t)=e^{2t}$$

$$p_2'=p_1+0p_2=e^{2t},\ p_2(0)=0\longrightarrow p_2(t)-p_2(0)=\int_0^te^{2\tau}d\tau=\frac12e^{2t}-\frac12$$

$$\Rightarrow e^{At}=M_0p_1+M_1p_2=e^{2t}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac12(e^{2t}-1)\begin{bmatrix}-1&1\\1&-1\end{bmatrix}$$

$$=\begin{bmatrix}\frac12e^{2t}+\frac12&\frac12(e^{2t}-1)\\\frac12(e^{2t}-1)&\frac12e^{2t}+\frac12\end{bmatrix}$$

A general soln of $x'=Ax$ is given for some vector $c=\begin{bmatrix}c_1\\c_2\end{bmatrix}$:

$$x(t)=e^{At}c=\begin{bmatrix}\frac12e^{2t}+\frac12&\frac12(e^{2t}-1)\\\frac12(e^{2t}-1)&\frac12e^{2t}+\frac12\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

$$=c_1\begin{bmatrix}\frac12(e^{2t}+1)\\\frac12(e^{2t}-1)\end{bmatrix}+c_2\begin{bmatrix}\frac12(e^{2t}-1)\\\frac12(e^{2t}+1)\end{bmatrix}$$

(ii) $x'=\overset{A}{\overbrace{\begin{bmatrix}2&1\\-1&4\end{bmatrix}}}x$

$\hookrightarrow$ by Putzer: $e^{At}=e^{3t}\begin{bmatrix}1-t&t\\-t&t+1\end{bmatrix}$

$\hookrightarrow$ gen soln is

$$x(t)=c_1e^{3t}\begin{bmatrix}1-t\\-t\end{bmatrix}+c_2e^{3t}\begin{bmatrix}t\\t+1\end{bmatrix}$$

(iii) $x'=\overset{A}{\overbrace{\begin{bmatrix}-1&-6\\1&4\end{bmatrix}}}x$

$\hookrightarrow$ by Putzer, $e^{At}=e^t\begin{bmatrix}3-2e^t&6-6e^t\\e^t-1&3e^t-2\end{bmatrix}$

$\hookrightarrow$ gen soln is

$$x(t)=c_1e^t\begin{bmatrix}3-2e^t\\e^t-1\end{bmatrix}+c_2e^t\begin{bmatrix}6-6e^t\\3e^t-2\end{bmatrix}$$

(iv) $x'=\overset{A}{\overbrace{\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}}}x$

$\hookrightarrow$ by Putzer, $e^{At}=e^t\begin{bmatrix}1&t&0\\0&1&0\\0&t&1\end{bmatrix}$

$\hookrightarrow$ gen soln given by some $c=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$ by

$$x(t)=e^{At}c=c_1e^t\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2e^t\begin{bmatrix}t\\1\\t\end{bmatrix}+c_3e^t\begin{bmatrix}0\\0\\1\end{bmatrix}$$

2.36 variation of constants (for constant A)

$$\begin{cases}x'=Ax+b(t)\\x(t_0)=x_0\end{cases}\Rightarrow x(t)=\Phi(t)\Phi^{-1}(t_0)x_0+\Phi(t)\int_{t_0}^t\Phi^{-1}(s)b(s)ds$$

ϕ is fund matrix

(i) $\begin{cases}x'=\overset{A}{\begin{bmatrix}1&0\\0&2\end{bmatrix}}x+\underset{b(t)}{\begin{bmatrix}1\\2\end{bmatrix}}\\x(0)=\begin{bmatrix}1\\1\end{bmatrix}\quad t_0=0\end{cases}$

Soln: Here, e-pairs of A are $\begin{cases}\lambda_1=1,\ v_1=\begin{bmatrix}1\\0\end{bmatrix}\\\lambda_2=2,\ v_2=\begin{bmatrix}0\\1\end{bmatrix}\end{cases}$

$\Rightarrow$ gen soln of $x'=Ax$ is

$$x(t)=c_1\begin{bmatrix}e^t\\0\end{bmatrix}+c_2\begin{bmatrix}0\\e^{2t}\end{bmatrix}$$

$$\Downarrow$$

Fund matrix is

$$\Phi(t)=\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}\longrightarrow\phi^{-1}(t)=\begin{bmatrix}e^{-t}&0\\0&e^{-2t}\end{bmatrix}\qquad\phi^{-1}(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

var of constants

$$\Rightarrow x(t)=\underbrace{\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}}_{\phi(t)}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{\phi^{-1}(t_0)}\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{x_0}+\underbrace{\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}}_{\phi(t)}\int_0^t\underbrace{\begin{bmatrix}e^{-s}&0\\0&e^{-2s}\end{bmatrix}}_{\phi^{-1}(s)}\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{b(s)}ds$$

$$=\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}\int_0^t\begin{bmatrix}e^{-s}\\2e^{-2s}\end{bmatrix}ds$$

$$=\begin{bmatrix}e^t\\e^{2t}\end{bmatrix}+\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}\begin{bmatrix}\int_0^te^{-s}ds\\2\int_0^te^{-2s}ds\end{bmatrix}$$

$=-e^{-s}\Big|_0^t=-e^{-t}-(-1)=1-e^{-t}$

$=2\left[-\frac12e^{-2s}\right]_0^t=-\left[e^{-2t}-1\right]=1-e^{-2t}$

$$=\begin{bmatrix}e^t\\e^{2t}\end{bmatrix}+\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}\begin{bmatrix}1-e^{-t}\\1-e^{-2t}\end{bmatrix}$$

$$=\begin{bmatrix}e^t\\e^{2t}\end{bmatrix}+\begin{bmatrix}e^t-1\\e^{2t}-1\end{bmatrix}=\begin{bmatrix}2e^t-1\\2e^{2t}-1\end{bmatrix}$$

(ii) $x'=\begin{bmatrix}2&0\\0&3\end{bmatrix}x+\begin{bmatrix}e^{2t}\\e^{3t}\end{bmatrix}$, $x(0)=\begin{bmatrix}1\\2\end{bmatrix}$

$\hookrightarrow$ fund. matrix

$$\phi(t)=\begin{bmatrix}e^{2t}&0\\0&e^{3t}\end{bmatrix}\longrightarrow\phi^{-1}(t)=\begin{bmatrix}e^{-2t}&0\\0&e^{-3t}\end{bmatrix}\text{ and }\phi^{-1}(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\Rightarrow x(t)=\phi(t)\phi^{-1}(0)x_0+\phi(t)\int_0^t\phi^{-1}(s)b(s)ds$$

$$=\underbrace{\begin{bmatrix}e^{2t}&0\\0&e^{3t}\end{bmatrix}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}_{=\begin{bmatrix}1\\2\end{bmatrix}}}_{=\begin{bmatrix}e^{2t}\\2e^{3t}\end{bmatrix}}+\begin{bmatrix}e^{2t}&0\\0&e^{3t}\end{bmatrix}\underbrace{\int_0^t\underbrace{\begin{bmatrix}e^{-2s}&0\\0&e^{-3s}\end{bmatrix}\begin{bmatrix}e^{2t}\\e^{3t}\end{bmatrix}}_{=\begin{bmatrix}1\\1\end{bmatrix}}dt}_{=\begin{bmatrix}t\\t\end{bmatrix}}$$

$$=\begin{bmatrix}e^{2t}\\2e^{3t}\end{bmatrix}+\underbrace{\begin{bmatrix}e^{2t}&0\\0&e^{3t}\end{bmatrix}\begin{bmatrix}t\\t\end{bmatrix}}_{\begin{bmatrix}te^{2t}\\te^{3t}\end{bmatrix}}=\begin{bmatrix}e^{2t}(1+t)\\e^{3t}(2+t)\end{bmatrix}$$

(iii) $x'=\begin{bmatrix}2&1\\0&2\end{bmatrix}x+\begin{bmatrix}e^{2t}\\te^{2t}\end{bmatrix}$, $x(0)=\begin{bmatrix}-1\\1\end{bmatrix}$

$\hookrightarrow$ fund matrix

$$\phi(t)=\begin{bmatrix}e^{2t}&te^{2t}\\0&e^{2t}\end{bmatrix}\longrightarrow\phi^{-1}(t)=\begin{bmatrix}e^{-2t}&-te^{-2t}\\0&e^{-2t}\end{bmatrix}$$

and $\phi^{-1}(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$x(t)=\phi(t)\phi^{-1}(0)x_0+\phi(t)\int_0^t\phi^{-1}(s)b(s)ds$$

$$=\underbrace{\begin{bmatrix}e^{2t}&te^{2t}\\0&e^{2t}\end{bmatrix}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}}_{=\begin{bmatrix}-1\\1\end{bmatrix}}}_{\begin{bmatrix}-e^{2t}+te^{2t}\\e^{2t}\end{bmatrix}}+\begin{bmatrix}e^{2t}&te^{2t}\\0&e^{2t}\end{bmatrix}\underbrace{\int_0^t\underbrace{\begin{bmatrix}e^{-2s}&-se^{-2s}\\0&e^{-2s}\end{bmatrix}\begin{bmatrix}e^{2s}\\se^{2s}\end{bmatrix}}_{=\begin{bmatrix}1-s^2\\s\end{bmatrix}}ds}_{=\begin{bmatrix}t-\frac{t^3}{3}\\\frac{t^2}{2}\end{bmatrix}}$$

$$=\begin{bmatrix}-e^{2t}+te^{2t}\\e^{2t}\end{bmatrix}+\underbrace{\begin{bmatrix}e^{2t}&te^{2t}\\0&e^{2t}\end{bmatrix}\begin{bmatrix}t-\frac{t^3}{3}\\\frac{t^2}{2}\end{bmatrix}}_{=\begin{bmatrix}te^{2t}-\frac{t^3}{3}e^{2t}+\frac{t^3}{2}e^{2t}\\\frac{t^2}{2}e^{2t}\end{bmatrix}}$$

$\frac12-\frac13=\frac36-\frac26=\frac16$

$$=\begin{bmatrix}-e^{2t}+te^{2t}\\e^{2t}\end{bmatrix}+\begin{bmatrix}te^{2t}+\frac16t^3e^{2t}\\\frac{t^2}{2}e^{2t}\end{bmatrix}$$

$$=\begin{bmatrix}e^{2t}(-1+2t+\frac16t^3)\\e^{2t}(1+\frac{t^2}{2})\end{bmatrix}$$

(iv) $x'=\begin{bmatrix}0&0&0\\0&2&1\\0&0&2\end{bmatrix}x+\begin{bmatrix}t\\0\\0\end{bmatrix}$, $x(0)=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\hookrightarrow$ fund matrix

$$\phi(t)=\begin{bmatrix}1&0&0\\0&e^{2t}&te^{2t}\\0&0&e^{2t}\end{bmatrix}\Rightarrow\phi^{-1}(t)=\begin{bmatrix}1&0&0\\0&e^{-2t}&-te^{-2t}\\0&0&e^{-2t}\end{bmatrix}$$

$$\phi^{-1}(0)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$x(t)=\phi(t)\phi^{-1}(0)x_0+\phi(t)\int_0^t\phi^{-1}(s)b(s)ds$$

$$=\underbrace{\begin{bmatrix}1&0&0\\0&e^{2t}&te^{2t}\\0&0&e^{2t}\end{bmatrix}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}_{=\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\begin{bmatrix}1\\e^{2t}+te^{2t}\\e^{2t}\end{bmatrix}}+\underbrace{\begin{bmatrix}1&0&0\\0&e^{2t}&te^{2t}\\0&0&e^{2t}\end{bmatrix}\underbrace{\int_0^t\underbrace{\begin{bmatrix}1&0&0\\0&e^{-2s}&-se^{-2s}\\0&0&e^{-2s}\end{bmatrix}\begin{bmatrix}s\\0\\0\end{bmatrix}}_{=\begin{bmatrix}s\\0\\0\end{bmatrix}}ds}_{\begin{bmatrix}t^2/2\\0\\0\end{bmatrix}}}_{=\begin{bmatrix}t^2/2\\0\\0\end{bmatrix}}$$

$$=\begin{bmatrix}1+\frac{t^2}{2}\\e^{2t}+te^{2t}\\e^{2t}\end{bmatrix}$$

2.40 Determine stability of trivial soln:

(i) $x'=\begin{bmatrix}-1&3\\2&-2\end{bmatrix}x$

$\hookrightarrow$ evals: $\lambda_1=-4$, $\lambda_2=1$

$\downarrow$ Thm 2.47

unstable b/c of λ_2!

(ii) $x'=\begin{bmatrix}5&8\\-7&-10\end{bmatrix}x$

$\hookrightarrow$ evals: $\lambda_1=-3$, $\lambda_2=-2$

$\downarrow$

asymptotically stable (all evals have real part <0)

(iii) $x'=\begin{bmatrix}0&2&1\\-2&0&-1\\0&0&-1\end{bmatrix}x$

$\hookrightarrow$ evals $\left.\begin{matrix}\lambda_1=2i\\\lambda_2=-2i\\\lambda_3=-1\end{matrix}\right\}\longrightarrow$ stable but not asy. stable b/c of λ_1 and λ_2

2.46 Find matrix norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ and the corresponding μ

for

$$A=\begin{bmatrix}-1&1\\2&-2\end{bmatrix}$$

$$\|A\|_1=\max_{1\le j\le2}\sum_{i=1}^2|a_{ij}|=\max\{|-1|+|2|,|1|+|-2|\}=\max\{3,3\}=3$$

$$\|A\|_\infty=\max_{1\le i\le2}\sum_{j=1}^2|a_{ij}|=\max\{|-1|+|1|,|2|+|-2|\}=\max\{2,4\}=4$$

and

$$\mu_{\|\cdot\|_1}(A)=\max_{1\le j\le2}\left\{a_{jj}+\sum_{i=1,i\ne j}^2|a_{ij}|\right\}=\max\{\underbrace{-1+|2|}_{a_{11}+|a_{21}|},-2+|1|\}=\max\{1,3\}=3$$

$a_{11}+|a_{21}|$

$a_{22}+|a_{12}|$

$$\mu_{\|\cdot\|_\infty}(A)=\max_{1\le i\le2}\left\{a_{ii}+\sum_{j=1,j\ne i}^2|a_{ij}|\right\}=\max\{-1+|2|,-2+|1|\}=\max\{1,-1\}=1$$

2.51 Find Log of ....

(a) $A=\begin{bmatrix}1&0\\0&e\end{bmatrix}\longrightarrow$ evals: $\lambda_1=1$, $v_1=\begin{bmatrix}1\\0\end{bmatrix}$; $\lambda_2=e$, $v_2=\begin{bmatrix}0\\1\end{bmatrix}$

Here,

$$J=\begin{bmatrix}1&0\\0&e\end{bmatrix}\longrightarrow\operatorname{Log}(J)=\begin{bmatrix}\ln(1)&0\\0&\ln(e)\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$D=[v_1\ v_2]=\begin{bmatrix}1&0\\0&1\end{bmatrix}\longrightarrow D^{-1}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\Rightarrow\operatorname{Log}(A)=D\operatorname{Log}(J)D^{-1}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

(b) $B=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\longrightarrow$ epairs $\lambda_1=2$, $v_1=\begin{bmatrix}1\\0\end{bmatrix}$; $\lambda_2=-1$, $v_2=\begin{bmatrix}0\\1\end{bmatrix}$

angle π ⇒ Arg(−1) = π

for $z\in\mathbb{C}$,

$$\operatorname{Log}(z)=\ln(|z|)+i\operatorname{Arg}(z)$$

$$J=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\longrightarrow\operatorname{Log}(J)=\begin{bmatrix}\operatorname{Log}(2)&0\\0&\operatorname{Log}(-1)\end{bmatrix}=\begin{bmatrix}\ln(2)&0\\0&i\pi\end{bmatrix}$$

$$D=[v_1\ v_2]=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I\longrightarrow D^{-1}=I$$

$$\Rightarrow\operatorname{Log}(B)=D\operatorname{Log}(J)D^{-1}=I\begin{bmatrix}\ln(2)&0\\0&\pi i\end{bmatrix}I=\begin{bmatrix}\ln(2)&0\\0&\pi i\end{bmatrix}$$

(c) $C=\begin{bmatrix}4&2\\-1&1\end{bmatrix}\longrightarrow$ epairs $\lambda_1=3$, $v_1=\begin{bmatrix}-2\\1\end{bmatrix}$; $\lambda_2=2$, $v_2=\begin{bmatrix}-1\\1\end{bmatrix}$

$$J=\begin{bmatrix}3&0\\0&2\end{bmatrix}\Rightarrow\operatorname{Log}(J)=\begin{bmatrix}\ln(3)&0\\0&\ln(2)\end{bmatrix}$$

$$D=[v_1\ v_2]=\begin{bmatrix}-2&-1\\1&1\end{bmatrix}\longrightarrow D^{-1}=\begin{bmatrix}-1&-1\\1&2\end{bmatrix}$$

$$\operatorname{Log}(C)=D\operatorname{Log}(J)D^{-1}=\begin{bmatrix}-2&-1\\1&1\end{bmatrix}\begin{bmatrix}\ln(3)&0\\0&\ln(2)\end{bmatrix}\begin{bmatrix}-1&-1\\1&2\end{bmatrix}$$

$$=\begin{bmatrix}-2\ln(3)&-\ln(2)\\\ln(3)&\ln(2)\end{bmatrix}\begin{bmatrix}-1&-1\\1&2\end{bmatrix}$$

$$=\begin{bmatrix}2\ln(3)-\ln(2)&2\ln(3)-2\ln(2)\\-\ln(3)+\ln(2)&-\ln(3)+2\ln(2)\end{bmatrix}$$

(d) $D=\begin{bmatrix}3&1\\-1&1\end{bmatrix}$

$\hookrightarrow$ epairs

$\lambda=2$, $v=\begin{bmatrix}-1\\1\end{bmatrix}\longleftarrow$ soln of $(A-\lambda I)v=0$

$\lambda=2$, $w=\begin{bmatrix}-1\\0\end{bmatrix}\longleftarrow$ soln of $(A-\lambda I)w=v$

Since e-value repeated, we take

$$J=\begin{bmatrix}2&1\\0&2\end{bmatrix}\longrightarrow\operatorname{Log}(J)=\begin{bmatrix}\ln(2)&\frac12\\0&\ln(2)\end{bmatrix}$$

$$D=[v\ w]=\begin{bmatrix}-1&-1\\1&0\end{bmatrix}\longrightarrow D^{-1}=\begin{bmatrix}0&1\\-1&-1\end{bmatrix}$$

$$\operatorname{Log}(D)=D\operatorname{Log}(J)D^{-1}$$

$$=\begin{bmatrix}-1&-1\\1&0\end{bmatrix}\begin{bmatrix}\ln(2)&\frac12\\0&\ln(2)\end{bmatrix}\begin{bmatrix}0&1\\-1&-1\end{bmatrix}$$

$$=\begin{bmatrix}-\ln(2)&-\frac12-\ln(2)\\\ln(2)&\frac12\end{bmatrix}\begin{bmatrix}0&1\\-1&-1\end{bmatrix}$$

$$=\begin{bmatrix}\frac12+\ln(2)&-\cancel{\ln(2)}+\frac12+\cancel{\ln(2)}\\-\frac12&\ln(2)-\frac12\end{bmatrix}$$

Graduate Students

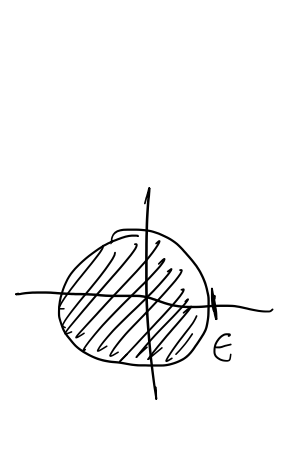
2.38 Let $\epsilon>0$. Graph $\{\vec{x}\in\mathbb{R}^2:\|\vec{x}\|<\epsilon\}$ when

(i) $\|\cdot\|$ is Euclidean norm

$$\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|=\sqrt{x^2+y^2}$$

$$\Rightarrow\sqrt{x^2+y^2}<\epsilon\longrightarrow x^2+y^2<\epsilon\longrightarrow$$

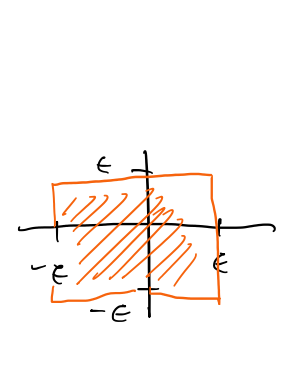
x^2 is increasing on $[0,\infty)$



(ii) $\|\cdot\|$ is max norm

$$\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|=\max\{|x|,|y|\}$$

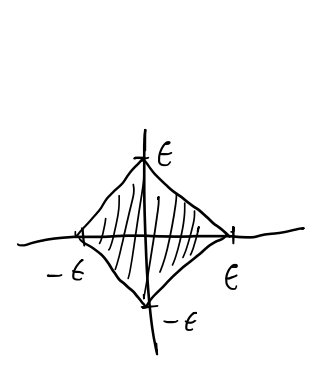
$$\Rightarrow\|\vec{x}\|<\epsilon\longleftrightarrow\max\{|x|,|y|\}<\epsilon\longrightarrow$$



(iii) $\|\cdot\|$ is traffic norm

$$\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|=|x|+|y|$$

$$\Rightarrow\|\vec{x}\|<\epsilon\longleftrightarrow|x|+|y|<\epsilon\longrightarrow$$



2.39 Prove: If A has eigenvalue w/ positive real part, then there is a soln of $x'=Ax$ so that $\|x(t)\|\to\infty$ as $t\to\infty$.

Proof: Let A have an eigenvalue λ with $\operatorname{Re}(\lambda)>0$, say $\lambda=\alpha+\beta i$ with $\alpha>0$, with e-vector $\begin{bmatrix}\gamma\\\delta\end{bmatrix}$. Then

$$x(t)=e^{(\alpha+i\beta)t}\begin{bmatrix}\gamma\\\delta\end{bmatrix}\text{ is a solution.}$$

Take $\|\cdot\|$ as, say, the Euclidean norm. Then,

$$\|x\|=\sqrt{\gamma^2e^{2(\alpha+i\beta)t}+\delta^2e^{2(\alpha+i\beta)t}}$$

$$=\underset{\to\infty\text{ as }t\to\infty}{e^{\alpha t}}\ \underset{\text{sines + cosines}}{e^{i\beta t}}\ \underbrace{\sqrt{\gamma^2+\delta^2}}_{\text{constant}}\longrightarrow\infty\text{ as }t\to\infty,$$

completing the proof. ▨