HW4 MTH 416/616 Spring 2025

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#2.6 Prove if one of $x_1, \dots, x_k$ is identically zero, then $\{x_1, \dots, x_k\}$ is linearly dependent.

Proof: Suppose WLOG $x_1 = 0$.

Consider

$$c_1 x_1 + c_2 x_2 + \dots + c_k x_k = 0$$

Let $c_1 = 1$ and $c_2 = c_3 = \dots = c_k = 0$. Then

$$0 c_1 = 0$$

So we have a nontrivial soln that yields zero vector.

Thus $\{x_1, \dots, x_k\}$ is linearly dependent.

#2.7 Prove fncts x, y are linearly dependent iff they are constant multiples of each other.

Proof: ($\longrightarrow$) If, say, x is the zero function, then $\{x, y\}$ is dependent (by #2.6) and

$$x = 0y.$$

So assume neither x nor y is the zero function. Then because $\{x, y\}$ is dependent $\exists$ nonzero c_1, c_2 so that for all t

$$c_1 x(t) + c_2 y(t) = 0$$

Then $x(t) = -\frac{c_2}{c_1} y(t)$, completing the proof in this direction.

($\longleftarrow$) Suppose $x = Cy$. If $C = 0$, then $x = 0$ and $\{x, y\}$ is dependent by #2.6.

If $C \neq 0$, then consider

$$c_1 x + c_2 y = 0$$

and choose $c_1 = 1$ and $c_2 = -C$ to get

$$x - Cy = 0$$
$$Cy - Cy = 0$$
$$0 = 0 \checkmark$$

Thus we see that $\{x, y\}$ is dependent. ∎

#2.18 Show characteristic eqt of any 2x2 constant matrix is

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

Use this to find chr eqt for

(i) $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$,

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and

(iii) $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

Soln: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, so chr eqt is

$$0 = det(A - \lambda I) = det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$
$$= (a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12}$$
$$= a_{11}a_{22} - a_{22}\lambda - a_{11}\lambda + \lambda^2 - a_{21}a_{12}$$
$$= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12})$$
$$= \lambda^2 - tr(A)\lambda + det(A)$$

(i) $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

$tr(A) = 1+3 = 4$ $det(A) = 3 - (-8) = 11$

$\therefore \lambda^2 - 4\lambda + 11 = 0$

(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$tr(A) = 1+4 = 5$ $det(A) = 4 - 6 = -2$

$\therefore \lambda^2 - 5\lambda - 2 = 0$

(iii) $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$

$tr(A) = 3+0 = 3$ $det(A) = 0 - 4$

$\therefore \lambda^2 - 3\lambda - 4 = 0$

#2.19 Solve

(i) $x' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x$

$tr(A) = 6$, $det(A) = 8 - 3 = 5$

Soln: Find evals: use #2.18 to get

$$\lambda^2 - 6\lambda + 5 = 0$$
$$(\lambda - 5)(\lambda - 1) = 0$$
$$\lambda = 5, 1$$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

evecter for $\lambda = 5$

$$\begin{bmatrix} 2-5 & 1 \\ 3 & 4-5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -3v_1 + v_2 = 0$

$v_2 = 3v_1$

$\Rightarrow v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} v_1$

take $v_1 = 1$

$\Rightarrow$ eigenpair $\lambda = 5$, $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

evecter for $\lambda = 1$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 + v_2 = 0$

$v_1 = -v_2$

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} v_1$

eigenpair $\Rightarrow \lambda = 1$, $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ soln to the DE is

$$x(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(ii) $x' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} x$ $A - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$

Soln: chr eqt is

$$\lambda^2 - \lambda - 6 = 0$$
$$(\lambda - 3)(\lambda + 2) = 0$$
$$\lambda = 3, -2$$

evecters

$\lambda = 3$

$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

$\to -v_1 + 2v_2 = 0$

$v_1 = -2v_2 \to v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} v_2$

$\to \lambda = 3$, $v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda = -2$

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

$\to 4v_1 + 2v_2 = 0$

$v_2 = -2v_1 \to v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} v_1$

$\to \lambda = -2$, $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow$ soln is

$$x(t) = c_1 e^{3t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(iii) didn't solve b/c repeated e-value here!

#2.20 We just solve (iii) because the others are similar:

$$x' = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x$$

Soln: Can't use #2.18 b/c not 2x2

find e-vals:

$$0 = det \begin{bmatrix} 1-\lambda & 2 & 1 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$
$$= (1-\lambda) det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} - 2 det \begin{bmatrix} -2 & 0 \\ 0 & 3-\lambda \end{bmatrix} + 1 det \begin{bmatrix} -2 & 1-\lambda \\ 0 & 0 \end{bmatrix}$$
$$= (1-\lambda)[(1-\lambda)(3-\lambda) - 0] - 2[-6 + 2\lambda] + 0$$
$$= (1-\lambda)^2(3-\lambda) - 4(\lambda - 3)$$
$$(1-\lambda)^2(3-\lambda) + 4(3-\lambda) = 0$$
$$(3-\lambda)[(1-\lambda)^2 + 4] = 0$$

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$3 - \lambda = 0$, $\lambda = 3$

$1 - 2\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm \frac{4i}{2} = 1 \pm 2i$$

evectors

for $\lambda = 3$:

$$\begin{bmatrix} -2 & 2 & 1 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2v_1 + 2v_2 + v_3 = 0 \to 2v_2 + 2v_2 + v_3 = 0$

$-2v_1 - 2v_2 = 0 \to v_1 = -v_2$

$4v_2 = -v_3$

$v_3 = -4v_2$

$\Rightarrow v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} v_2$

$\Rightarrow$ eigenpair: $\lambda = 3$, $v = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

for $\lambda = 1 + 2i$

$$\begin{bmatrix} 1-(1+2i) & 2 & 1 \\ -2 & 1-(1+2i) & 0 \\ 0 & 0 & 3-(1+2i) \end{bmatrix} = \begin{bmatrix} -2i & 2 & 1 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{bmatrix}$$

So

$$\begin{bmatrix} -2i & 2 & 1 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$
$-2iv_1 + 2v_2 + v_3 = 0 \to -2i(-iv_2) + 2v_2 + 0 = 0 \to -2v_2 - 2v_2 = 0$, $0 = 0 \checkmark$

$-2v_1 - 2iv_2 = 0 \to v_1 = -iv_2$

$(2-2i)v_3 = 0 \to v_3 = 0$

So

$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} v_2 \to$

$\Rightarrow$ eigenpair $\lambda = 1 + 2i$, $v = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} i$

So this eigenpair gives soln

$$e^{(1+2i)t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} i\right)$$
$$= e^t[\cos(2t) + i\sin(2t)]\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} i\right)$$
$$= e^t\left(\cos(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i\cos(2t)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + i\sin(2t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i^2\sin(2t)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$i^2 = -1$

$$= e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + i e^t\begin{bmatrix} -\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$$

"u" "v" → two indep solns!

So we have general soln

$$x(t) = c_1 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + c_2 e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + c_3 e^t\begin{bmatrix} -\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$$

#2.22 Find fund. matrix for

$$x' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} x$$

Soln: didn't solve b/c repeated e-values

(if distinct, find all 3 solns then make them cols of matrix to get the fund. matrix)

Grad students

#2.12 Find two functions indep on $\mathbb{R}$ but dependent on $(0, \infty)$

Soln: Consider $f_1(x) = |x|$ and $f_2(x) = x$

Then on $(0, \infty)$, $f_1 = f_2 \Rightarrow$ dependent

On $\mathbb{R}$, independent b/c

$$c_1 f_1 + c_2 f_2 = 0$$
$$c_1|x| + c_2 x = 0$$

$x = 1$: $c_1 + c_2 = 0 \to c_1 = -c_2$

$x = -1$: $c_1 - c_2 = 0 \to c_1 = c_2$

only $c_1 = c_2 = 0$ is possible

$\Rightarrow$ Independent on $\mathbb{R}$

#2.16 If λ_0 eigenvalue of A, find eigenvalue of ...

(i) A^T

Since $\left\{ \begin{array}{l} det\, X^T = det\, X \\ (X+Y)^T = X^T + Y^T \end{array} \right\}$, we see that since $\lambda = \lambda_0$ solves $det(A - \lambda I) = 0$ and

$$A^T - \lambda I = (A - \lambda I)^T$$

and so $\lambda = \lambda_0$ satisfies

$$det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I) = 0$$

(ii) A^n

we know $\exists v$ s.t. $Av = \lambda_0 v$, so

$$A^2 v = \lambda_0 (Av) = \lambda_0^2 v$$
$$\vdots$$
$$A^n v = \lambda_0^n v$$

Thus λ_0^n is an eigenvalue of A^n

(iii) A^{-1} ← we know $\lambda_0 \neq 0$ otherwise A wouldn't be invertible!

from $Av = \lambda_0 v$, we get

$$v = \lambda_0 A^{-1} v, \text{ so}$$
$$A^{-1} v = \frac{1}{\lambda_0} v$$

Thus $\frac{1}{\lambda_0}$ is an eigenvalue of A

#2.25 Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Proof: The chr eqt for A is (immediate from #2.18)

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Compute

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}$$

Now compute

$$A^2 - (a+d)A + (ad - bc)I$$
$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ca + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$