HW3 MTH 416/616 Spring 2025

Friday, February 7, 2025 7:31 PM

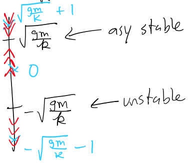
Ch. 1 #1.21 $mv' = mg - kv^2$

$$\downarrow$$

$$v' = \boxed{g - \frac{k}{m}v^2 \overset{\text{set}}{=} 0}$$

$$\downarrow$$

$$v^2 = \frac{gm}{k} \longrightarrow v = \pm\sqrt{\frac{gm}{k}}$$



$\sqrt{\frac{gm}{k}}+1$
$\sqrt{\frac{gm}{k}}$ ← asy stable
0
$-\sqrt{\frac{gm}{k}}$ ← unstable
$-\sqrt{\frac{gm}{k}}-1$

test point v	$g - \frac{k}{m}v^2 =$
$\sqrt{\frac{gm}{k}}+1$	$g - \frac{k}{m}\left(\sqrt{\frac{gm}{k}}+1\right)^2 = g - \frac{k}{m}\left[\frac{gm}{k}+2\sqrt{\frac{gm}{k}}+1\right]$ $= \overset{0}{\cancel{g-g}} - 2\frac{k}{m}\sqrt{\frac{gm}{k}} - \frac{k}{m} < 0$ ⊖
0	$g - \frac{k}{m}(0^2) = g$ ⊕
$-\sqrt{\frac{gm}{k}}-1$	$g - \frac{k}{m}\left(-\sqrt{\frac{gm}{k}}-1\right)^2 = g - \frac{k}{m}\left(\frac{gm}{k}+2\sqrt{\frac{gm}{k}}+1\right)$ $= \cancel{g-g} - \frac{2k}{m}\sqrt{\frac{gm}{k}} - \frac{k}{m} < 0$ ⊖

(ii) Solve $\begin{cases} v' = g - \frac{k}{m}v^2 \\ v(0) = v_0 \end{cases}$

and show as $t \to \infty$ the soln $\to \sqrt{\frac{gm}{k}}$

Soln: $\frac{dv}{dt} = g - \frac{k}{m}v^2$

$\downarrow$ separation of variables

$$(\star)\quad \boxed{\underbrace{\int \frac{1}{g - \frac{k}{m}v^2}}_{\text{rewrite as}}\,dv = \int 1\,dt = t + C}$$

difference of squares

$$\int \frac{1}{(\sqrt{g})^2 - \left(\sqrt{\frac{k}{m}}\,v\right)^2}\,dv = \int \frac{1}{\left(\sqrt{g} - \sqrt{\frac{k}{m}}\,v\right)\left(\sqrt{g} + \sqrt{\frac{k}{m}}\,v\right)}\,dv$$

partial fractions

$$\frac{1}{(\sqrt{g}-\sqrt{\frac{k}{m}}v)(\sqrt{g}+\sqrt{\frac{k}{m}}v)} = \frac{A}{\sqrt{g}-\sqrt{\frac{k}{m}}v} + \frac{B}{\sqrt{g}+\sqrt{\frac{k}{m}}v}$$

$\Downarrow$ mult by denom of LHS

$$1 = A\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right) + B\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)$$

if $v = \frac{-\sqrt{g}}{\sqrt{\frac{k}{m}}}$, then coeff of $A = 0$ so we get

$$1 = B\left(\sqrt{g}+\cancel{\sqrt{\frac{k}{m}}}\frac{\sqrt{g}}{\cancel{\sqrt{\frac{k}{m}}}}\right)$$

$$1 = 2\sqrt{g}\,B$$

$$\downarrow$$

$$B = \frac{1}{2\sqrt{g}}$$

$v = \frac{+\sqrt{g}}{\sqrt{\frac{k}{m}}}$

$$1 = 2\sqrt{g}\,A$$

$$\downarrow$$

$$A = \frac{1}{2\sqrt{g}}$$

Therefore,

$$\int \frac{1}{(\sqrt{g})^2-(\sqrt{\frac{k}{m}}v)^2}\,dv = \int \frac{1/2\sqrt{g}}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\,dv + \int \frac{1/2\sqrt{g}}{\sqrt{g}+\sqrt{\frac{k}{m}}v}\,dv$$

$u = \sqrt{g}-\sqrt{\frac{k}{m}}v$

$du = -\sqrt{\frac{k}{m}}\,dv$

$-\sqrt{\frac{m}{k}}\,du = dv$

and

$w = \sqrt{g}+\sqrt{\frac{k}{m}}v$

$dw = \sqrt{\frac{k}{m}}\,dv$

$\sqrt{\frac{m}{k}}\,dw = dv$

$$= \left(\frac{1}{2\sqrt{g}}\right)\left(-\sqrt{\frac{m}{k}}\right)\int \frac{1}{u}\,du + \left(\frac{1}{2\sqrt{g}}\right)\left(\sqrt{\frac{m}{k}}\right)\int \frac{1}{w}\,dw$$

$$= -\frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right) + \frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left(\sqrt{g}+\sqrt{\frac{k}{m}}v\right)$$

Therefore, (A) becomes

$$\frac{1}{2}\sqrt{\frac{m}{gk}}\left[\ln\left(\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\right)\right] = t + C$$

$$\ln\left(\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v}\right) = 2\sqrt{\frac{gk}{m}}\,t + \tilde{C}\ ;\ \tilde{C} = 2\sqrt{\frac{gk}{m}}\,C$$

$$\frac{\sqrt{g}+\sqrt{\frac{k}{m}}v}{\sqrt{g}-\sqrt{\frac{k}{m}}v} = \hat{C}e^{2\sqrt{\frac{gk}{m}}t}\ ;\ \hat{C} = e^{\tilde{C}}$$

$$\downarrow$$

$$\sqrt{g}+\sqrt{\frac{k}{m}}v = \left(\sqrt{g}-\sqrt{\frac{k}{m}}v\right)\hat{C}e^{2\sqrt{\frac{gk}{m}}t}$$

$$\left(\sqrt{\frac{k}{m}}+\hat{C}\sqrt{\frac{k}{m}}e^{2\sqrt{\frac{gk}{m}}t}\right)v = -\sqrt{g}+\sqrt{g}\,\hat{C}e^{2\sqrt{\frac{gk}{m}}t}$$

Thus,

$$v(t) = \frac{\sqrt{g}\left(-1+\hat{C}e^{2\sqrt{\frac{gk}{m}}t}\right)}{\sqrt{\frac{k}{m}} + \hat{C}\sqrt{\frac{k}{m}}\, e^{2\sqrt{\frac{gk}{m}}t}}$$

Now initial condition gives

$$v_0 \underset{\substack{\uparrow\\ \text{given}}}{=} v(0) = \frac{-\sqrt{g}+\sqrt{g}\,\hat{C}}{\sqrt{\frac{k}{m}} + \hat{C}\sqrt{\frac{k}{m}}}$$

$$\downarrow$$

$$(v_0)\left(\sqrt{\frac{k}{m}}+\hat{C}\sqrt{\frac{k}{m}}\right) = -\sqrt{g}+\sqrt{g}\,\hat{C}$$

$$v_0\sqrt{\frac{k}{m}} + \hat{C}v_0\sqrt{\frac{k}{m}} = -\sqrt{g}+\sqrt{g}\,\hat{C}$$

$$\hat{C}\left(v_0\sqrt{\frac{k}{m}}-\sqrt{g}\right) = -\sqrt{g} - v_0\sqrt{\frac{k}{m}}$$

Thus, $\boxed{\hat{C} = \dfrac{-\sqrt{g}-v_0\sqrt{\frac{k}{m}}}{v_0\sqrt{\frac{k}{m}}-\sqrt{g}}}$ ← (messy, so I'll keep writing $\tilde{C}$ in what follows)

Now compute

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{\sqrt{g}\left(-1+\hat{C}e^{2\sqrt{\frac{gk}{m}}t}\right)}{\sqrt{\frac{k}{m}} + \hat{C}\sqrt{\frac{k}{m}}\, e^{2\sqrt{\frac{gk}{m}}t}} = \frac{\infty}{\infty}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{t\to\infty} \frac{(\sqrt{g})\cancel{\hat{C}}\left(\cancel{2\sqrt{\frac{gk}{m}}}\right)\cancel{e^{2\sqrt{\frac{gk}{m}}t}}}{\cancel{\hat{C}}\sqrt{\frac{k}{m}}\left(\cancel{2\sqrt{\frac{gk}{m}}}\right)\cancel{e^{2\sqrt{\frac{gk}{m}}t}}}$$

$$= \lim_{t\to\infty}\frac{\sqrt{g}}{\sqrt{\frac{k}{m}}} = \sqrt{\frac{mg}{k}}$$ ← this is precisely the equilibrium we found earlier!

#1.23

(i) $$N' = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{logistic growth}} \underbrace{-\lambda N}_{\substack{\text{decreases rate of change by}\\ \text{an amount proportional to}\\ \text{the population size}}}, \quad \lambda > 0$$

$$\underbrace{\hphantom{\text{decreases rate of change by an amount}}}_{\text{this represents "harvesting"}}$$

(ii) if $\lambda \overset{0<r-\lambda}{<} r$, compute + analyze equilibria set RHS = 0 :

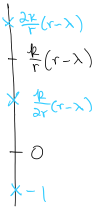
$$rN\left(1-\frac{N}{k}\right)-\lambda N = 0$$

$$N\left[r-\frac{rN}{k}-\lambda\right]=0$$

$$\swarrow \quad \text{or} \quad \searrow$$

$$N=0 \qquad \underbrace{(r-\lambda)}_{>0}-\frac{rN}{k}=0$$

$$\Rightarrow N=\frac{k}{r}(r-\lambda)$$



test point N =	$rN(1-\frac{N}{k})-\lambda N$
$\frac{2k}{r}(r-\lambda)$	$\cancel{r}\left(\frac{2k}{\cancel{r}}\right)(r-\lambda)\left(1-\frac{1}{k}\left(\frac{2k}{r}(r-\lambda)\right)\right)-\lambda\left(\frac{2k}{r}(r-\lambda)\right)$ $=2k(r-\lambda)\left(1-\frac{2}{r}(r-\lambda)\right)-\lambda\left(\frac{2k}{r}(r-\lambda)\right)$ $=2k(r-\lambda)-\frac{4k}{r}(r-\lambda)-\lambda\left(\frac{2k}{r}(r-\lambda)\right)$ $=(r-\lambda)\left[2k-\frac{4k}{r}-\frac{2k\lambda}{r}\right]$ $=2k(r-\lambda)\left[1-\frac{2}{r}-\frac{\lambda}{r}\right]$ $=\frac{2k(r-\lambda)}{r}\left[r-2-\lambda\right]$

For potential energy function F, we know

$$F'(N)=-\left[rN\left(1-\frac{N}{k}\right)-\lambda N\right]$$

so

$$F''(N)=-\left[r\left(1-\frac{N}{k}\right)+rN\left(-\frac{1}{k}\right)-\lambda\right]$$

and thus

$$F''(0)=-\left[r+0-\lambda\right]<0 \quad (\text{not a local min})$$

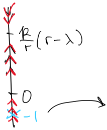
$$F''\left(\frac{k}{r}(r-\lambda)\right)=-\left[r\left(1-\frac{1}{\cancel{k}}\left[\frac{\cancel{k}}{r}(r-\lambda)\right]\right)-\frac{\cancel{r}}{\cancel{k}}\left[\frac{\cancel{k}}{\cancel{r}}(r-\lambda)\right]-\lambda\right]$$

$$=-\left[r-(r-\lambda)-(r-\lambda)-\lambda\right]$$

$$=-\left[\cancel{r}-\cancel{r}+\lambda-r+\cancel{\lambda}-\cancel{\lambda}\right]$$

$$=-(\lambda-r)=r-\lambda>0$$

Therefore asymptotically stable at this equilibrium



$$F''(-1)=-\left[-r\left(1+\frac{1}{k}\right)+\frac{r}{k}-\lambda\right]$$

$$=r+\cancel{\frac{r}{k}}-\cancel{\frac{r}{\lambda}}-\lambda$$

$$=r-\lambda>0 \quad \oplus$$

(iii) We have equilibria

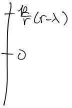
$$N=0 \quad \text{and} \quad N=\frac{k}{r}(r-\lambda)$$

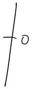
$\underline{r>\lambda}$ $\qquad$ $\underline{r=\lambda}$ $\qquad$ $\underline{r<\lambda}$

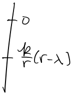
asy stable equilibrium at $\frac{k}{r}(r-\lambda)$

$N=0$
↑
equilibria merge

means $r-\lambda<0$ so







transcritical bifurcation at bifurcation pt $r=\lambda$

#1.25

(i) $x' = \lambda - 4 - x^2$

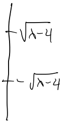
Soln: Set $\lambda - 4 - x^2 = 0$

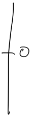
$$x = \pm\sqrt{\lambda - 4}$$

$\lambda - 4 > 0$

$\lambda - 4 = 0$

$\lambda - 4 < 0$
no equilibria





↑
bifurcation point when $\lambda=4$

Saddle-node bifurcation

(ii) $x' = x^3(\lambda - x)$

Soln: set $x^3(\lambda - x) = 0$

$$\swarrow \text{ OR } \searrow$$

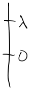
$x^3 = 0$ $\quad$ $\lambda = x$

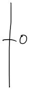
$\swarrow$

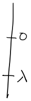
$x = 0$

bifurcation pt at $\lambda = 0$ ↙

$\lambda > 0$ $\quad$ $\lambda = 0$ $\quad$ $\lambda < 0$







transcritical bifurcation

#1.26 $x' = \frac{1}{10}x(x-25)$

Soln: killing rats equation at rate λ:

$$x' = \frac{1}{10}x(x-25) - \lambda x$$

↑ "harvesting"

equilibria:

set $\frac{1}{10}x(x-25) - \lambda x = 0$

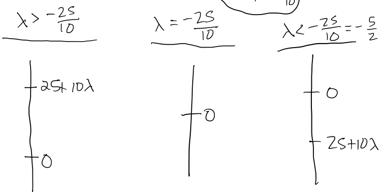
$$\frac{x^2}{10} - \frac{25x}{10} - \lambda x = 0$$

$$x^2 - 25x - 10\lambda x = 0$$

$$x^2 - (25+10\lambda)x = 0$$

$$\begin{tikzcd}
 & & & \lambda > -\frac{25}{10} \\
 & x(x-(25+10\lambda)) = 0 \arrow[dl, "\text{or}"] \arrow[dr] & & 25+10\lambda > 0 \arrow[u] \\
x = 0 & & x = 25+10\lambda \arrow[ur, ">0"] \arrow[r, "=0"] \arrow[dr, "<0"'] & \begin{matrix} 25+10\lambda = 0 \\ \lambda = -\frac{25}{10} \end{matrix} \\
 & & & \begin{matrix} 25+10\lambda < 0 \\ \lambda < -\frac{25}{10} \end{matrix}
\end{tikzcd}$$

transcritical bifurcation at $\lambda = -\frac{25}{10}$


$\lambda > -\frac{25}{10}$
$25+10\lambda$
0
$\lambda = -\frac{25}{10}$
0
$\lambda < -\frac{25}{10} = -\frac{5}{2}$
0
$25+10\lambda$

here $F''(x) = -\frac{d}{dx}\left[\frac{1}{10}x^2 - \frac{25}{10}x - \lambda x\right]$

$$= -\left[\frac{1}{5}x - \frac{25}{10} - \lambda\right]$$

and so $F''(0) = \frac{25}{10} + \lambda$

$\lambda > -\frac{25}{10} \rightarrow F''(0) > 0 \rightarrow$ local min

$\lambda = -\frac{25}{10} \rightarrow F''(0) = 0 \rightarrow$ unstable

$\lambda < -\frac{25}{10} \rightarrow F''(0) < 0 \rightarrow$ unstable

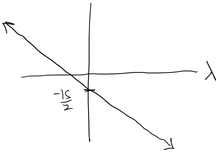
$$-5 - \frac{5}{2} = \frac{-10}{2} - \frac{5}{2} = -\frac{15}{2}$$

$$F''(25+10\lambda) = -\left[\frac{1}{5}(25+10\lambda) - \frac{25}{10} - \lambda\right]$$

$$= -5 - 2\lambda - \frac{25}{10} - \lambda$$

$$= -3\lambda - \frac{15}{2}$$

we have



$F''(25+10\lambda) > 0$

whenever

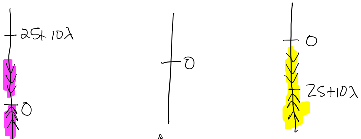
$-3\lambda - \frac{15}{2} > 0$

$-3\lambda > \frac{15}{2}$

$\lambda < \frac{-15}{6} = \frac{-5}{2}$

bifurcation pt

$\lambda > \frac{-25}{10}$ | $\lambda = \frac{-25}{10}$ | $\lambda < \frac{-25}{10} = -\frac{5}{2}$



can't physically happen !!

if $\lambda > \frac{-25}{10}$

then the rat population will go extinct

unstable at 0

Ch.2 #2.2 Let $M : \mathbb{A} \to \mathbb{A}$ — ctn functions on I

$$(Mx)(t) = \int_a^t x(s)\,ds$$

Show M is linear operator.

Soln: Let $f, g \in \mathbb{A}$ and $\alpha, \beta \in \mathbb{R}$ and compute

$$\big(M(\alpha f + \beta g)\big)(t) = \int_a^t (\alpha f + \beta g)(s)\,ds$$

$$= \alpha \int_a^t f(s)\,ds + \beta \int_a^t g(s)\,ds$$

$$= \alpha (Mf)(t) + \beta (Mg)(t)$$

§2.3 dependent or independent?

(i) $\left\{ \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$

Soln: Consider equation

$$c_1\begin{bmatrix}-4\\4\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\-2\end{bmatrix}+c_3\begin{bmatrix}2\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{cases}-4c_1+c_2+2c_3=0\\ 4c_1-c_2-2c_3=0\\ c_1-2c_2+c_3=0\end{cases}$$

can solve $\Longrightarrow$ w/ Gaussian elimination

$$\begin{bmatrix}-4&1&2\\4&-1&-2\\1&-2&1\end{bmatrix}\overset{r_2^*=r_2+r_1}{\sim}\begin{bmatrix}-4&1&2\\0&0&0\\1&-2&1\end{bmatrix}$$

$\uparrow$

this matrix is not invt'bl b/c of the row of 0's

$\Rightarrow$ ∞-many solns exist

$\Rightarrow$ linearly dependent

(ii) $\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\3\\2\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$

Soln: faster way: compute

$$\det\left(\begin{bmatrix}2&-1&1\\1&3&-2\\-1&2&1\end{bmatrix}\right)=2\det\left(\begin{bmatrix}3&-2\\2&1\end{bmatrix}\right)$$

$$-(-1)\det\left(\begin{bmatrix}1&-2\\-1&1\end{bmatrix}\right)+\det\left(\begin{bmatrix}1&3\\-1&2\end{bmatrix}\right)$$

$$=2(3+4)+(1-2)+(2+3)$$

$$=14-1+5\neq 0$$

$\Rightarrow$ independent

(iii) $\left\{\begin{bmatrix}2\\1\\-1\\-2\end{bmatrix},\begin{bmatrix}-1\\2\\1\\1\end{bmatrix},\begin{bmatrix}1\\13\\2\\-1\end{bmatrix}\right\}$

Soln: can't use det trick here, so set up

$$\begin{cases}2c_1-c_2+c_3=0 \longrightarrow \boxed{c_1=\dfrac{c_2-c_3}{2}}\\ c_1+2c_2+13c_3=0 \longrightarrow \dfrac{c_2-c_3}{2}+2c_2+13c_3=0\\ -c_1+c_2+2c_3=0\end{cases}$$

$-2c_1+c_2-c_3=0$

$\frac{5}{2}c_2+\frac{25}{2}c_3=0$

$\boxed{c_2=-\frac{25}{2}c_3\left(\frac{2}{5}\right)=-5c_3}$

$\boxed{c_1=\frac{-5c_3-c_3}{2}=-3c_3}$

$-2(-3c_3)+(-5c_3)-c_3=0$

$6c_3-5c_3-c_3=0$

$\downarrow$

$0=0$

$3c_3-5c_3+2c_3=0 \to 0=0$

(no contribution to soln)

⇒ we only get $c_1=-3c_3$ ← free var!

$c_2=-5c_3$

⇒ dependent — for example $c_3=1,\ c_2=-5,\ c_1=-3$ is a nonzero soln

#2.5 lin. dep or lin indep on $I=\mathbb{R}$?

(i) $x_1(t)=4\sin(t),\ x_2(t)=7\sin(-t)$

Soln: Dependent: $x_2(t)=-7\sin(t)$, ← sine is "odd" function

so

$c_1(4\sin(t))-7c_2\sin(t)=0$

has nontrivial soln

$c_1=\frac{1}{4}\quad c_2=\frac{1}{7}$

(ii) $x_1(t)=2\sin(2t),\ x_2(t)=-3\cos(2t),\ x_3(t)=8$

Soln: $c_1x_1+c_2x_2+c_3x_3=0$

$2c_1\sin(2t)-3c_2\cos(2t)+8c_3=0$

$t=0$

$0-3c_2+8c_3=0$

$c_2=\frac{8}{3}c_3 \to c_3=\frac{3}{8}c_1$

$t=\frac{\pi}{4}$

$2c_1\sin\left(\frac{2\pi}{4}\right)-3c_2\cos\left(\frac{2\pi}{4}\right)+8c_3=0$

$2c_1-0+8c_3=0$

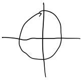
$\boxed{c_3=\frac{-2}{8}c_1=-\frac{1}{4}c_1}$

$t=-\frac{\pi}{4}$

$2c_1\sin\left(-\frac{2\pi}{4}\right)-3c_2\cos\left(-\frac{2\pi}{4}\right)+8c_3=0$

$-2c_1-0+8c_3=0$

$\boxed{c_3=\frac{2c_1}{8}=\frac{c_1}{4}}$



$-\frac{c_1}{4}=c_3=\frac{c_1}{4}$

$\downarrow$

$c_1=-c_1 \to c_1=0$

$\Rightarrow c_3=0$

$\Rightarrow c_2 = 0$

Thus independent

(iii) $x_1(t) = te^{3t}$, $x_2(t) = e^{-2t}$

$$\begin{tikzcd}
& c_1te^{3t} + c_2e^{-2t} = 0 \arrow[dl, "t=0"'] \arrow[dr, "t=1"] & \\
c_2 = 0 \arrow[rr] & & c_1e^3 + 0 = 0 \\
& & C_1 = 0
\end{tikzcd}$$

∴ independent

(iv) $x_1(t) = te^{3t}$, $x_2(t) = e^{3t+4}$

Soln: consider

$$\begin{tikzcd}
& c_1te^{3t} + c_2e^{3t+4} = 0 \arrow[dl, "t=0"'] \arrow[dr, "t=1"] & & \\
c_2e^4 = 0 \arrow[d] & & c_1e^3 + 0 = 0 \arrow[r] & c_1 = 0 \\
c_2 = 0 \arrow[urr] & & &
\end{tikzcd}$$

∴ independent

(v) $x_1(t) = \sin(t)$, $x_2(t) = \cos(t)$, $x_3(t) = \sin(t + \frac{\pi}{4})$

$$\overset{\text{sum of angles identity}}{=} \sin(t)\cos(\tfrac{\pi}{4}) + \cos(t)\sin(\tfrac{\pi}{4})$$

$$= \frac{\sqrt{2}}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t)$$

Then we get dependent because

$$-\frac{\sqrt{2}}{2}x_1(t) - \frac{\sqrt{2}}{2}x_2(t) + x_3(t) = 0$$

Grad student problems

#1.24 $x' = \lambda - x + \frac{4x^2}{1+x^2}$

(i) for $\lambda = 1$: set $1 - x + \frac{4x^2}{1+x^2} \overset{\text{set}}{=} 0$

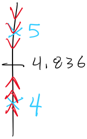
$$(1-x)(1+x^2) + 4x^2 = 0$$

$$1 + x^2 - x - x^3 + 4x^2 = 0$$

$$-x^3+5x^2-x+1=0$$

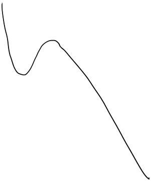
$\swarrow$ one real soln (two complex)

$$x \approx 4.836$$

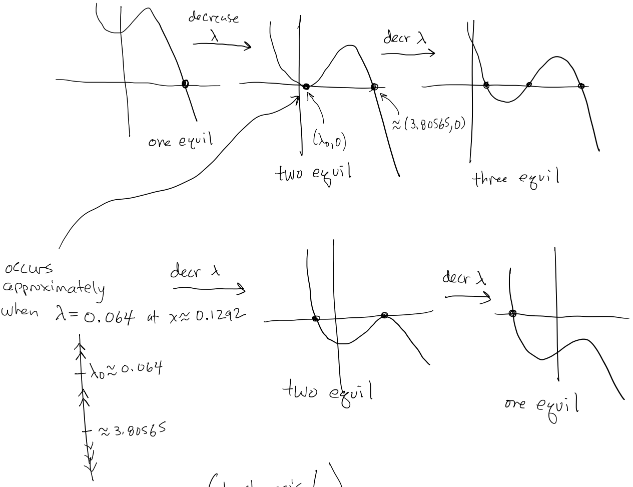


test value $x=$	$1-x+\frac{4x^2}{1+x^2}=$
5	$1-5+\frac{100}{26}<0$ ⊖
4	$1-4+\frac{64}{17}>0$ ⊕

(ii) graph of $\lambda - x + \frac{4x^2}{1+x^2}$ looks like



choice of λ shifts it vertically



occurs approximately when $\lambda = 0.064$ at $x \approx 0.1292$

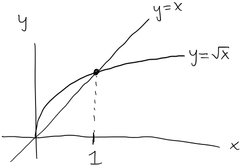
(hysteresis!)

#1.28

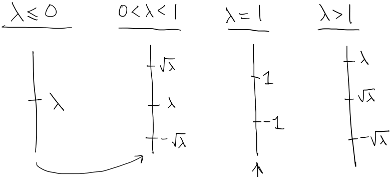
(i) $x' = (\lambda - x)(\lambda - x^2) \overset{\text{set}}{=} 0$

OR

$x = \lambda$ $\quad$ $x = \pm\sqrt{\lambda}$



(no $\sqrt{\lambda}$ equil b/c complex)



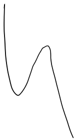
pitchfork bifurcation at $\lambda = 0$

transcritical bifurcation here at $\lambda = 1$

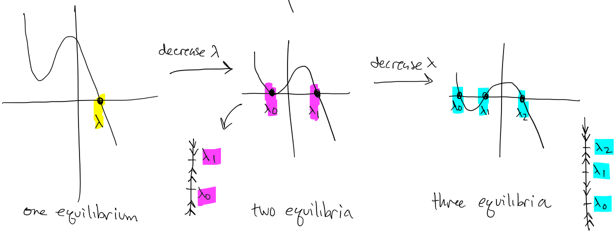
($\sqrt{\lambda}$ passes thru to other side of λ)

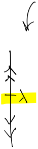
(ii) $x' = \lambda - 12 + 3x - x^3$

general shape of RHS:

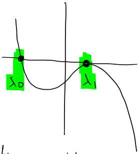


and λ moves it vertically, so we get

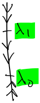




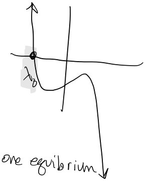
decr. λ →



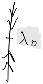
two equilibria



decr λ →



one equilibrium

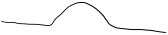


(hysteresis!)

(iii) $x' = \lambda - \frac{x^2}{1+x^2} \overset{\text{set}}{=} 0$

$\lambda - \frac{x^2}{1+x^2}$

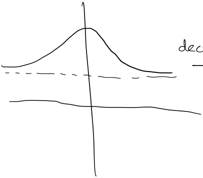
looks like



where λ moves it vertically

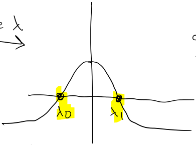
saddle-node

occurs at $\lambda = 0$

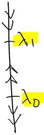


no equilibria

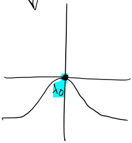
decrease λ →



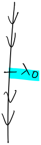
two equilibria

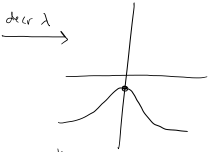


decr λ →



one equilibrium



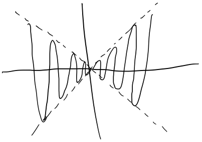


no equilibria

⊕

$$(iv)\ x' = x\,\overset{=0\text{ whenever } x = n\pi}{\widetilde{\sin(x)}} + \lambda$$

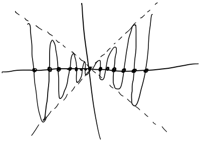
x sin(x) looks like



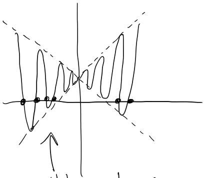
for every λ, we end up with ∞-many equilibria

λ = 0

λ > 0

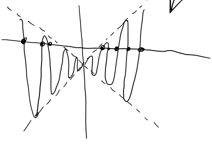


equilibria at $x = n\pi,\ n \in \mathbb{Z}$



equilibria always found "far away enough from 0"

λ < 0



Complicated to analyze → don't worry about it ☺

#2.1 ★ chr eqt for $y''+ay'+by=0$

found by letting $y=e^{rt} \rightarrow (r^2+ar+b)e^{rt}=0$

$$r^2+ar+b=0$$

$y^{(2)}+p_1(t)y'+p_0(t)y=0$

★ matrix form of $y''+ay'+by=0$ is

by p.21:

$$A(t)=\begin{bmatrix}0 & 1\\ -p_0(t) & -p_1(t)\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix}}_{\text{Companion matrix}}$$

Compute

$$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda & 1\\ -b & -a-\lambda\end{bmatrix}$$

$$=(-\lambda)(-a-\lambda)-(-b)(1)$$

$$=\lambda^2+a\lambda+b\overset{set}{=}0$$

Same equation!!

#2.4 $\underbrace{\phi_1(t)=\ln(t),\ \phi_2(t)=\ln(t^2)}_{\text{linearly dependent because}}$, $t\in(0,\infty)$

$$\phi_2(t)=\ln(t^2)=2\ln(t)\text{, so}$$

$$c_1\phi_1(t)+c_2\phi_2(t)=0$$

$$\Rightarrow (c_1+2c_2)\ln(t)=0$$

So we get a nontrivial soln $c_1=-2c_2$ for arbitrary $c_2\neq 0$