HW2 MTH 416.616 Spring 2025

Monday, February 3, 2025 12:12 AM

Ch 1 #1.12 Show the equilibrium $x^*=0$ for the DE $x'=0$ is (stable) but not (asymptotically stable).

<u>Soln</u>:

stable means: $\forall \epsilon>0\ \exists \delta>0$ if $|x_1-x^*|<\delta$, then soln $\phi(\cdot,x_1)$ exists and $|\phi(t,x_1)-x_0|<\epsilon$

asymptotically stable means: $\exists \delta_0$ s.t. $|x_1-x_0|<\delta_0$ implies $\lim_{t\to\infty}\phi(t,x_1)=x^*$

First solve $\begin{cases} x'=0 \\ x(0)=x_1 \end{cases}$: integrate the DE to get $x=C$ for some constant C. Apply initial cond to get

$$x_1 \underset{\text{given}}{=} x(0)=C \rightarrow C=x_1.$$

Thus soln to IVP is $x(t)=x_1$.

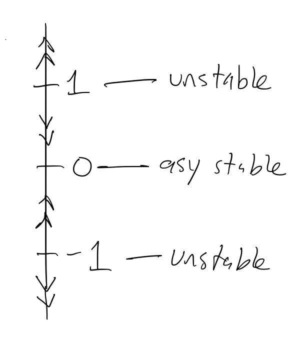
<u>Show $x^*=0$ is stable</u>: Let $\epsilon>0$ and choose $\delta=\epsilon$. As long as $|x_1|<\delta=\epsilon$, we get $|\phi(t,x_1)-x^*|=|x_1|<\epsilon$. Thus it is stable.

<u>Show not asymptotically stable</u>

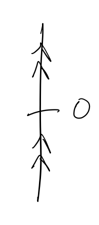
Solns are constant, so $\lim_{t\to\infty}\phi(t,x_1)=\lim_{t\to\infty}x_1=x_1\neq 0=x^*$. ($x_1\neq 0$)

#1.13 Determine stability of equilibrium pts of... (see #1.7 from HW1)

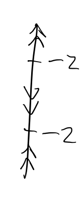
(a) $x'=-x+x^3$



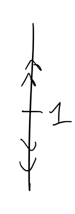
(b) $x'=x^4$



(c) $x'=x^2+4x+2$



(d) $x'=x^3-3x^2+3x-1$



#1.14 yam in 200°C oven at $t=0$

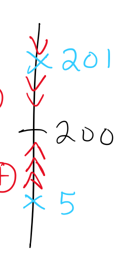
$T(t)$ — temp of yam at time t

$$T'=-k(T-200),\quad k>0$$

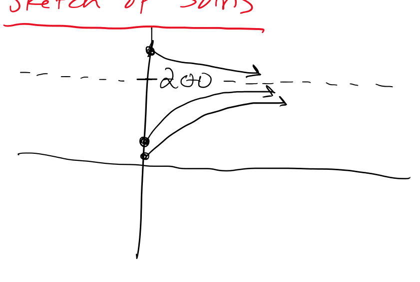
Set $=0$: $-k(T-200)=0$

$T=200$

test pt $T=$	$-k(T-200)=$
201	$-k(1)<0$ ⊖
5	$-k(-195)>0$ ⊕



→ Sketch of solns



#1.16 Find potential energy fnct (F) & use it to draw phase line diagram:

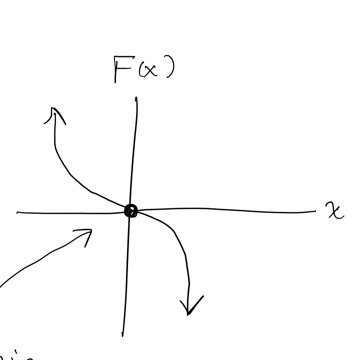
(a) $x'=x^2$

<u>Soln</u>: Here $x^2=0 \rightarrow x=0$ is equil pt

$$-F'(x)=x^2$$

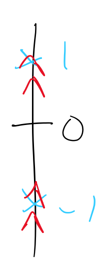
$$\rightarrow F'(x)=-x^2$$

$$\int_0^x \rightarrow F(x)=-\int_0^x t^2\,dt=-\frac{x^3}{3}$$



not a local min ⟹ not asymp stable

test pt x	x^2
1	$1>0$
-1	$1>0$



(b) $x'=\boxed{3x^2-10x+6\overset{\text{set}}{=}0}$

$$x=\frac{10\pm\sqrt{100-4(3)(6)}}{2(3)}=\frac{10}{6}\pm\frac{\sqrt{28}}{6}=\frac{5}{3}\pm\frac{\sqrt{7}}{3}$$

$318 \times 4 = 72$; $100-72=28$; $\sqrt{28}=2\sqrt{7}$

critical pts of f ⟷ zeros of right-hand side

<u>crit pts of F</u>:

$$F'(x)=-3x^2+10x-6\overset{\text{set}}{=}0$$

$$x=\frac{-10\pm\sqrt{100-4(-3)(-6)}}{(-3)(2)}=\frac{10}{6}\pm\frac{\sqrt{28}}{-6}=\frac{5}{3}\pm\frac{2\sqrt{7}}{-6}=\frac{5}{3}\mp\frac{\sqrt{7}}{3}\ \text{(same)}$$

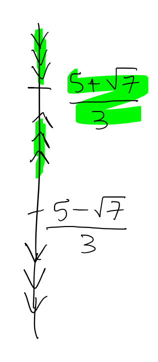
$24\times 3=72$; $100-72=28$

<u>extrema</u>

Compute $F''(x)=-6x+10$

$F''\left(\frac{5}{3}-\frac{\sqrt{7}}{3}\right)=-6\left(\frac{5-\sqrt{7}}{3}\right)+10=-10-2\sqrt{7}+10<0$ — not local min

$F''\left(\frac{5}{3}+\frac{\sqrt{7}}{3}\right)=-6\left(\frac{5+\sqrt{7}}{3}\right)+10=-10+2\sqrt{7}+10>0$ — local min (asy stable!)



(c) $x'=8x-4x^3$

$$-F'(x)=8x-4x^3$$

$$F'(x)=4x^3-8x\overset{\text{set}}{=}0 \rightarrow x(4x^2-8)=0$$

$x=0$ or $4x^2-8=0$, $x^2=2$, $x=\pm\sqrt{2}$

<u>extrema</u>

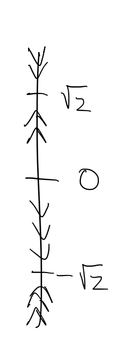
$F''(x)=12x^2-8$

$F''(0)=-8$

$F''(\sqrt{2})=24-8=16>0$ — local min

$F''(-\sqrt{2})=16>0$ — local min

⟹



(d) $x'=\frac{1}{x^2+1}$

<u>Soln</u>: $-F'(x)=\frac{1}{x^2+1}$

$F'(x)=\frac{-1}{x^2+1}\overset{\text{set}}{=}0 \rightarrow$ <u>no soln</u> ⟹ no equilibria!

$F(x)=-\arctan(x)$

<u>Graduate student problems</u>

1.11 Assume p,q periodic w/ period $T>0$. ($p(t+T)=p(t)$, $q(t+T)=q(t)$)

Prove: $x'=px+q$ has periodic soln w/ period T iff x has soln satisfying $x(0)=x(T)$

<u>Proof</u>: (→) Suppose x is a periodic soln w/ period T, i.e. $x(t+T)=x(t)$ for all t. Setting $t=0$ yields $x(T)=x(0)$, completing the proof in this direction.

(←) Suppose a solution x obeys $x(0)=x(T)$.
By Thm 1.6, we know soln (taking $t_0=0$) obeys

$$x(t)=e^{\int_0^t p(\tau)d\tau}\left[x_0+\int_0^t e^{-\int_0^s p(\tau)d\tau}q(s)\,ds\right]$$

Compute

$x(0)=x_0$ and

$$x(T)=e^{\int_0^T p(\tau)d\tau}\left[x_0+\int_0^T e^{-\int_0^s p(\tau)d\tau}q(s)\,ds\right]$$

By assumption, we have $x(0)=x(T)$, so

$$x_0=e^{\int_0^T p(\tau)d\tau}\left[x_0+\int_0^T e^{-\int_0^s p(\tau)d\tau}q(s)\,ds\right]$$

$$\downarrow$$

$$\left(1-e^{\int_0^T p(\tau)d\tau}\right)x_0=\left(e^{\int_0^T p(\tau)d\tau}\right)\int_0^T e^{-\int_0^s p(\tau)d\tau}q(s)\,ds$$

$$\downarrow$$

$$x_0=\frac{\left(e^{\int_0^T p(\tau)d\tau}\right)\int_0^T e^{-\int_0^s p(s)ds}q(s)\,ds}{1-e^{\int_0^T p(\tau)d\tau}}$$

This means the solution corresponding to this x_0 (if one exists) would be the only possible periodic solution.

Let x be soln to the IVP $\begin{cases} x'=px+q \\ x(0)=x_0 \end{cases}$, which we know is unique. We ask: is x periodic? To check it, consider the function $y(t)=x(t+T)$. Compute

$$y'(t)=x'(t+T)\underset{\substack{\text{since } x \text{ solves}\\ x'=px+q}}{=}p(t+T)x(t+T)+q(t+T)$$

$$=p(t)x(t+T)+q(t) \quad (p,q \text{ are } T\text{-periodic})$$

$$=p(t)y(t)+q(t). \quad (\text{def of } y)$$

At $t=0$, we see that $y(0)=x(T)=x(0)=x_0$ (assumption of this direction!)

Thus y solves $\begin{cases} y'=py+q \\ y(0)=x_0 \end{cases}$. Since the solution is unique and it solves the same DE as x, we must conclude $y(t)=x(t)$, i.e. by def of y,

$$x(t+T)=x(t),$$

completing the proof. ∎

#1.18 $x'=ax\ln\left(\frac{b}{x}\right)$, $x>0$, $a>0$, $b>0$

Find equilibria & its stability.

<u>Soln</u>: Set $ax\ln\left(\frac{b}{x}\right)=0$ (can't have $x=0$)

So must get

$$\ln\left(\frac{b}{x}\right)=0$$

$$\frac{b}{x}=e^0=1$$

$$\rightarrow \boxed{b=x}$$

Compute

$$\underbrace{-\frac{d}{dx}ax\ln\left(\frac{b}{x}\right)}_{F''}=-a\left[\ln\left(\frac{b}{x}\right)+\frac{x}{\left(\frac{b}{x}\right)}\frac{d}{dx}\left(\frac{b}{x}\right)\right] \quad \text{(chain rule)}$$

$$=-a\left[\ln\left(\frac{b}{x}\right)+\frac{x^2}{b}\left(\frac{-b}{x^2}\right)\right]$$

$$=-a\ln\left(\frac{b}{x}\right)+a$$

So at $x=b$ this becomes

$$F''(b)=-a\ln\left(\frac{b}{b}\right)+a=a>0$$

Thus asy stable equilibrium at $x=b$.