HW1 MTH 416.616 Spring 2025

Friday, January 31, 2025 8:46 AM

Ch. 1 #1.1 Interval of existence of soln to

$$\begin{cases} x' = (\cos t)x^2 \\ x(0) = 2 \end{cases}$$

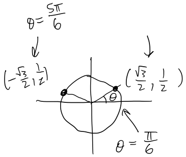
Soln: Solve this IVP with sep of variables:

$$\frac{dx}{dt} = \cos(t)\, x^2$$

$$\int x^{-2}\,dx = \int \cos(t)\,dt$$

$$-\frac{1}{x} = \sin(t) + C$$

$$x = \frac{-1}{\sin(t)+C}$$



$\theta = \frac{5\pi}{6}$
$(-\frac{\sqrt{3}}{2}, \frac{1}{2})$
$(\frac{\sqrt{3}}{2}, \frac{1}{2})$
θ
$\theta = \frac{\pi}{6}$

Apply initial cond:

$$\underset{\substack{\uparrow \\ \text{given}}}{2} = x(0) \underset{\substack{\uparrow \\ \text{computed}}}{=} \frac{-1}{0+C} \rightarrow \frac{1}{2} = -C \rightarrow C = -\frac{1}{2}$$

$$\Rightarrow x(t) = \frac{-1}{\sin(t) - \frac{1}{2}}$$

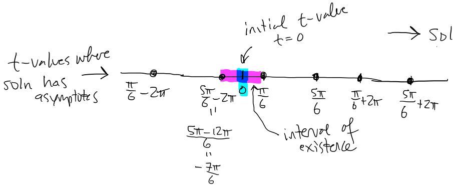
To find int of existence, consider where $\sin(t) - \frac{1}{2} = 0$

$$\rightarrow \sin(t) = \frac{1}{2}$$

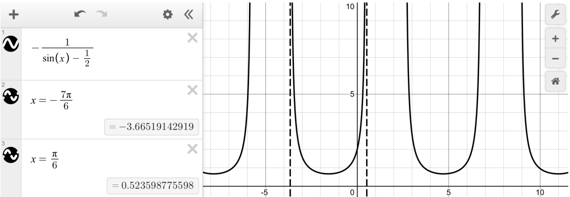
$$\rightarrow \text{Solns } t = \frac{\pi}{6} \pm 2\pi n,\ n \in \mathbb{Z}$$

or

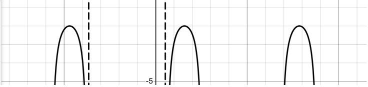
$$t = \frac{5\pi}{6} + 2\pi n,\ n \in \mathbb{Z}$$



$$\Rightarrow \text{interval of existence is } \left(-\frac{7\pi}{6}, \frac{\pi}{6}\right)$$



1. $-\frac{1}{\sin(x) - \frac{1}{2}}$
2. $x = -\frac{7\pi}{6}$ $= -3.66519142919$
3. $x = \frac{\pi}{6}$ $= 0.523598775598$



#1.2 Interval of existence of soln to

$$\begin{cases} x' = \frac{2tx^2}{1+t^2} \\ x(0) = x_0 \end{cases}$$

Soln: Solve w/ sep of vars:

$u = 1+t^2$

$du = 2t\,dt$

$$\int x^{-2}dx = 2\int \frac{t}{1+t^2}\,dt$$

$$-\frac{1}{x} = \int \frac{1}{u}\,du$$

$$-\frac{1}{x} = \ln(1+t^2)+C$$

$$x = \frac{-1}{\ln(1+t^2)+C}$$

Apply IC:

$$\underset{\substack{\uparrow \\ \text{given}}}{x_0} = x(0) = \frac{-1}{\ln(1)+C} = -\frac{1}{C}$$

$x_0 = 0$ ↙

NO SOLN HERE

↙

$x=0$ is a soln of the DE w/ interval of existence $(-\infty,\infty)$

$x_0 \neq 0$ ↘

$$\boxed{C = -\frac{1}{x_0}}$$

↓

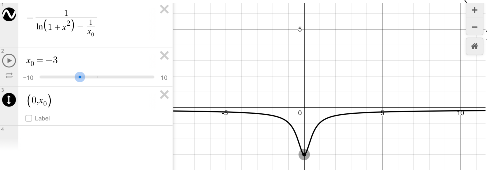
$$x(t) = \frac{-1}{\ln(1+t^2) - \frac{1}{x_0}} = \frac{-x_0}{x_0\ln(1+t^2) - 1}$$

Case 1: $x_0 = 0$

↳ soln is $x(t) = 0$ has int existence $(-\infty,\infty)$

Case 2: $x_0 < 0 \leftarrow x_0 = -|x_0|$

Here, denominator could be problem



$-\frac{1}{\ln(1+x^2)-\frac{1}{x_0}}$

$x_0 = -3$

$(0, x_0)$

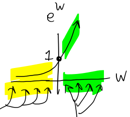
if

$x_0 \ln(1+t^2) - 1 = 0$

$\ln(1+t^2) = \frac{-1}{|x_0|}$

$1+t^2 = e^{-1/|x_0|}$

$t^2 = \underbrace{-1 + e^{-\frac{1}{|x_0|}}}$

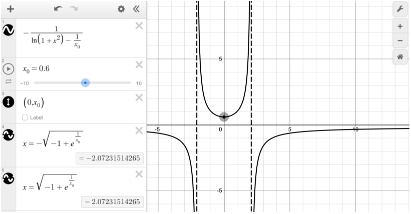


but this is negative since $e^{-\frac{1}{|x_0|}} < 1$, so

$-1+e^{-1/|x_0|} < 0$

So can't take $\sqrt{\ }$, so NO asymptotes

$\Rightarrow$ int of existence is $(-\infty, \infty)$



$-\frac{1}{\ln(1+x^2)-\frac{1}{x_0}}$

$x_0 = 0.6$

$(0, x_0)$

$x = -\sqrt{-1+e^{\frac{1}{x_0}}}$ $= -2.07231514265$

$x = \sqrt{-1+e^{\frac{1}{x_0}}}$ $= 2.07231514265$

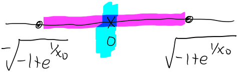
Case 3 $x_0 > 0$

denominator: $x_0 \ln(1+t^2) - 1 = 0$

$t^2 = \underbrace{-1 + e^{\frac{1}{x_0}}}_{>0}$

$e^{\frac{1}{x_0}} \geq 1$

$\Rightarrow t = \pm\sqrt{-1+e^{1/x_0}}$



$\Rightarrow$ int of existence is $\left(-\sqrt{-1+e^{1/x_0}}, \sqrt{-1+e^{1/x_0}}\right)$

#1.3 Show that the IVP $\begin{cases} x' = x^{1/3} \\ x(0) = 0 \end{cases}$ has ∞-many solns.

Explain why Thm 1.3 does not apply to give uniqueness.

Soln: First notice the constant function $x(t) = 0$ is a solution.

On the other hand, we use separation of variables technique to solve $x' = x^{1/3}$:

$$\frac{dx}{dt} = x^{1/3} \to \int x^{-1/3}\,dx = \int 1\,dt$$

$$\frac{x^{2/3}}{2/3} = t + C \to x^{2/3} = \frac{2}{3}t + \tilde{C};\quad \tilde{C} = \frac{2}{3}$$

$$\to x = \left(\frac{2}{3}t + \tilde{C}\right)^{3/2}$$

$$\to \underset{\substack{\uparrow \\ \text{given}}}{0 =} x(0) \underset{\substack{\uparrow \\ \text{computed}}}{=} \tilde{C}^{3/2} \to \tilde{C} = 0$$

$$\Rightarrow x(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2} \quad \text{is a soln}$$

That's two... to get ∞-many solns, glue them together: let $a \in [0,\infty)$ and let

new solns
$$x(t) = \begin{cases} 0, & t \le a \\ \left(\frac{2}{3}\right)^{3/2}(t-a)^{3/2}, & t > a \end{cases}$$

Is it a solution? let $t \in (-\infty, a]$,

Compute $x(t) = 0$ (since $t \le a$ for all $t \in (-\infty, a]$) ✓

Clearly it solves on $(-\infty, a)$ from earlier. Does it solve at $t \ge a$?

For $t > a$

$$x'(t) = \overbrace{\left(\frac{2}{3}\right)^{3/2}\left(\frac{3}{2}\right)}^{\frac{(2/3)^{3/2}}{2/3} =}(t-a)^{1/2} = \overbrace{\left(\frac{2}{3}\right)^{1/2}}(t-a)^{1/2}$$

match! so, equal

while

$$x(t)^{1/3} = \left[\left(\frac{2}{3}\right)^{3/2}(t-a)^{3/2}\right]^{1/3} = \left(\frac{2}{3}\right)^{1/2}(t-a)^{1/2}$$

For $t = a$

Must check that x is diff'bl at $t = a$, so compute

$$\lim_{h\to 0^+} \frac{x(a+h) - x(a)}{h} = \lim_{h\to 0^+} \left(\frac{2}{3}\right)^{3/2}\left[\frac{(a+h-a)^{3/2} - \cancelto{0}{(a-a)^{3/2}}}{h}\right]$$

$$= \left(\frac{2}{3}\right)^{3/2} \lim_{h\to 0^-} \frac{h^{3/2}}{h}$$

$$= \left(\frac{2}{3}\right)^{3/2} \lim_{h\to 0^-} \sqrt{h} = 0$$

while $$\lim_{h\to 0^-} \frac{x(a+h) - x(a)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$$

So we see also that x is diff'bl at 0 and $x'(0) = 0^{1/3} = 0$. ✓

Thm 1.3 does not apply to give uniqueness because for $f(x,t) = x^{1/3}$, $\frac{\partial f}{\partial x} = f_x(t,x) = \frac{1}{3}x^{-2/3}$, which is not continuous at $x = 0$.

#1.4 Let $\begin{cases} f: \mathbb{R}\times(0,\infty) \to \mathbb{R} \\ f(t,x) = x^{1/3} \end{cases}$ Show that $\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$ has a unique soln for any $(t_0,x_0) \in \mathbb{R}\times(0,\infty)$

Find max interval of existence for $(t_0, x_0) = (1,1)$.

Soln: Solve by separation of variables (as in #1.3) to get solution

$$x(t) = \left(\frac{2}{3}t + C\right)^{3/2}$$

Apply initial condition $x(t_0)=x_0$ $(t_0>0,\ x_0>0)$ to obtain

$$x_0 \underset{\substack{\uparrow\\ \text{given}}}{=} x(t_0) \underset{\substack{\uparrow\\ \text{computed}}}{=} \left(\tfrac{2}{3}t_0 + C\right)^{3/2}$$

$$x_0^{2/3} = \tfrac{2}{3}t_0 + C \longrightarrow C = x_0^{2/3} - \tfrac{2}{3}t_0$$

So the soln is

$$x(t) = \left(\tfrac{2}{3}t + x_0^{2/3} - \tfrac{2}{3}t_0\right)^{3/2}.$$

Apply initial condition $x(1)=1$ to get

$$1 \underset{\substack{\uparrow\\ \text{given}}}{=} x(1) \underset{\substack{\uparrow\\ \text{computed}}}{=} \left(\tfrac{2}{3}+C\right)^{3/2} \Rightarrow 1^{2/3} = \tfrac{2}{3} + C$$

$$\Rightarrow C = 1 - \tfrac{2}{3} = \tfrac{1}{3}$$

$\Rightarrow$ solution is

$$x(t) = \left(\tfrac{2}{3}t + \tfrac{1}{3}\right)^{3/2}$$

Because of the square root, we require $\frac{2}{3}t + 1 \geq 0$ to obtain a real-valued solution. Thus, $t \geq -\frac{3}{2}$, so the interval of existence is $(-\frac{3}{2}, \infty)$.

notice $f(t,x)$ exists for such t, so the soln is unique

#1.5 (i) Solve $\begin{cases} x' = 2x + e^{2t} \\ x(0) = 3 \end{cases}$

$$\boxed{\begin{matrix} p(t)=2 & x_0 = 3 \\ q(t) = e^{2t} & t_0 = 0 \end{matrix}}$$

Soln: Directly from Thm 1.6, we get

$$\int_{t_0}^{t} p(\tau)\,d\tau = \int_0^t 2\,d\tau = 2t,$$ so

$$x(t) = 3e^{2t} + e^{2t}\int_0^t \underbrace{(e^{-2s})(e^{2s})}_{=1}\,ds$$

$$= 3e^{2t} + te^{2t}$$

(ii) $\begin{cases} x' = 3x + 2e^{3t} \\ x(0) = 2 \end{cases}$

Soln: From Thm 1.6 with $\boxed{\begin{matrix} p(t)=3 & x_0 = 2 \\ q(t) = 2e^{3t} & t_0 = 0 \end{matrix}}$, we get

$$\int_{t_0}^{t} p(\tau)\,d\tau = \int_0^t 3\,d\tau = 3t,$$ so

$$x(t) = 2e^{3t} + e^{3t}\int_0^t e^{-3s}(2e^{3s})\,ds$$

$$= 2e^{3t} + 2e^{3t}\int_0^t 1\,ds$$

$$= 2e^{3t} + 2te^{3t}$$

(iii) $\begin{cases} x' = \tan(t)\,x + \sec(t) \\ x(0) = -1 \end{cases}$

Soln: Apply Thm 1.6 with $\boxed{\begin{matrix} p(t)=\tan(t) & x_0=-1 \\ q(t)=\sec(t) & t_0=0 \end{matrix}}$ to get

$$\int_{t_0}^{t} p(\tau)\,d\tau = \int_0^t \tan(\tau)\,d\tau = -\ln(\cos t) - \left(\underbrace{-\cancel{\ln}((\cos(0)))}_{=\ln(1)=0}\right)$$

$$= -\ln(\cos(t))$$

$\int \tan\tau\,d\tau$

$= \int \frac{\sin\tau}{\cos\tau}\,d\tau$

$u=\cos\tau \quad du=-\sin\tau\,d\tau$

$= -\int \frac{1}{u}\,du$

$= -\ln(u)+C$

$= -\ln(\cos\tau)+C$

Also calculate

$$e^{\int_{t_0}^{t} p(\tau)d\tau} = e^{\int_0^t \tan(\tau)d\tau} = e^{-\ln(\cos t)} = \frac{1}{e^{\ln(\cos t)}} = \frac{1}{\cos t} = \sec(t)$$

Therefore, $e^{-\int p} = \cos(t)$ and we get

$$x(t) = -\sec(t) + \sec(t)\int_0^t \underbrace{\cos(s)\sec(s)}_{=1}\,ds = -\sec(t) + t\sec(t)$$

$$= (t-1)\sec(t)$$

(iv) $\begin{cases} x' = \frac{2}{t}x + t \\ x(1) = 2 \end{cases}$

Soln: Apply Thm 1.6 with $\boxed{\begin{matrix} p(t)=\frac{2}{t} & t_0=1 \\ q(t)=t & x_0=2 \end{matrix}}$ to get

$$e^{\int_{t_0}^{t} p(\tau)d\tau} = e^{2\int_1^t \frac{1}{\tau}d\tau} = e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

and so $e^{-\int p} = \frac{1}{t^2}$. Thus,

$$x(t) = 2t^2 + t^2\int_1^t \underbrace{\frac{s}{s^2}}_{=\frac{1}{s}}\,ds = 2t^2 + t^2\ln(t)$$

#1.7 | Draw phase line diagrams for...

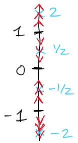
(i) $x' = -x + x^3$

Soln: Set RHS=0 to get $-x+x^3=0$

$x(-1+x^2)=0$

$\swarrow$ or $\searrow$

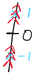
$x=0 \qquad x^2-1=0 \Leftrightarrow x=\pm 1$



test value $x=$	$-x+x^3=$
2	$-2+8=6$ ⊕
1/2	$-\frac{1}{2}+\frac{1}{8}$ ⊖
-1/2	$\frac{1}{2}-\frac{1}{8}$ ⊕
-2	$2-8$ ⊖

(ii) $x' = x^4$

Soln: Set RHS=0 to get $x^4 = 0 \Rightarrow x = 0$

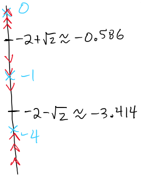


test value x	$x^4 =$
1	1 ⊕
-1	1 ⊕

(iii) $x' = x^2 + 4x + 2$

Soln: Solve $x^2 + 4x + 2 = 0$ $\qquad \sqrt{8} = 2\sqrt{2}$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2} = -2 \pm \frac{\sqrt{8}}{2} = -2 \pm \sqrt{2}$$



test value x	$x^2+4x+2=$
0	2 ⊕
-1	1-4+2 ⊖
-4	16-16+2 ⊕

(iv) $x' = x^3 - 3x^2 + 3x - 1$

Soln: Solve $x^3 - 3x^2 + 3x - 1 = 0$

Coeffs are similar! So this is

$(x-1)^3 = 0$

$x = 1$

recall Pascal Δ

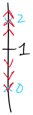
1
1 1
1 2 1
1 3 3 1

Binomial thm

$$(x-1)^3 = \sum_{k=0}^{3} \binom{3}{k} x^k (-1)^{3-k}$$

$$= \binom{3}{0} x^0 (-1)^3 + \binom{3}{1} x (-1)^2 + \binom{3}{2} x^2 (-1)^1 + \binom{3}{3} x^3 (-1)^0$$

$$= -1 + 3x - 3x^2 + x^3$$



test point x	$x^3-3x^2+3x-1=$
2	8-12+6-1 ⊕
0	-1 ⊖

Graduate student problems:

#1.8 If x nonzero soln to $x' = -rx + \frac{r}{K}$; $r, K > 0$. Then $N = \frac{1}{x}$ is a soln to $N' = rN\left(1 - \frac{N}{K}\right)$.

Soln: Calculate

$$N' = \frac{d}{dt}\left(\frac{1}{x}\right) = \frac{-x'}{x^2} = \frac{-\left(-rx + \frac{r}{K}\right)}{x^2}$$

$$= \frac{r}{x}\left(\frac{x - \frac{1}{K}}{x}\right)$$

$$= \frac{r}{x}\left(1 - \left(\frac{1}{x}\right)\left(\frac{1}{K}\right)\right)$$

$$= rN\left(1 - \frac{N}{K}\right)$$

#1.9 Bernoulli's eqt: $y' = p(t)y + q(t)y^\alpha$, $\alpha \neq 0, 1$

Show if y is nonzero soln of Bernoulli eqt, then $x = y^{1-\alpha}$ solves the equation $x' = (1-\alpha)p(t)x + (1-\alpha)q(t)$

Soln: Let $x = y^{1-\alpha}$, so
$$x' = (1-\alpha)y^{-\alpha}y'$$
$$= (1-\alpha)y^{-\alpha}\left[p(t)y + q(t)y^{\alpha}\right]$$
$$= (1-\alpha)y^{1-\alpha}\left[p(t) + q(t)y^{\alpha-1}\right] \qquad y^{\alpha-1} = \frac{1}{y^{1-\alpha}}$$
$$= (1-\alpha)x\left[p(t) + \frac{q(t)}{x}\right]$$
$$= (1-\alpha)p(t)x + (1-\alpha)q(t),$$
completing the proof. ∎

#1.10 Solve

(i) $y' = -\frac{1}{t}y + ty^2$

Soln: Let $x = y^{1-2} = \frac{1}{y}$. Then by #1.9, x solves
$$x' = (-1)\left(-\tfrac{1}{t}\right)x + (-1)t = \left(\tfrac{1}{t}\right)x + (-t)$$

Thm 1.6 with
$p(t) = \frac{1}{t}$ $t_0 = t_0$
$q(t) = -t$ $x_0 = x_0$
$$\Rightarrow e^{\int_{t_0}^{t} p(\tau)d\tau} = e^{\int_{t_0}^{t}\frac{1}{\tau}d\tau} = e^{\ln(t)-\ln(t_0)} = e^{\ln\left(\frac{t}{t_0}\right)} = \frac{t}{t_0}$$
and $e^{-\int p} = \frac{t_0}{t}$. So,
$$\Rightarrow x(t) = \frac{tx_0}{t_0} + \frac{t}{\cancel{t_0}}\int_{t_0}^{t}\frac{\cancel{t_0}}{s}(-s)\,ds = \frac{tx_0}{t_0} - t(t-t_0)$$
Therefore since $x = \frac{1}{y}$, we obtain soln
$$y(t) = \frac{1}{x(t)} = \frac{1}{\frac{tx_0}{y_0} - t(t-t_0)}$$

(ii) $y' = y + e^t x^2$

Soln: Let $x = y^{1-2} = \frac{1}{y}$. Then by #1.9, x solves
$$x' = (-1)(1)x + (-1)e^t = -x - e^t$$

Thm 1.6 with
$p(t) = -1$
$q(t) = -e^t$
$$\Rightarrow e^{\int_{t_0}^{t} -1\,d\tau} = e^{-(t-t_0)}$$
$$\Rightarrow x(t) = x_0e^{-(t-t_0)} + e^{-(t-t_0)}\int_{t_0}^{t}\underbrace{e^{s-t_0}(-e^{-s})}_{=-e^{-t_0}}ds$$
$$= x_0e^{-(t-t_0)} + e^{-(t-\cancel{t_0})}\cancel{e^{-t_0}}(t-t_0)$$
$$= x_0e^{-(t-t_0)} + e^{-t}(t-t_0)$$
Therefore
$$y(t) = \frac{1}{x(t)} = \frac{1}{x_0e^{-(t-t_0)} + e^{-t}(t-t_0)}$$

(iii) $y' = -\frac{1}{t}y + \frac{1}{ty^2}$

Soln: Since $\frac{1}{y^2} = y^{-2}$, let $x = y^{1-(-2)} = y^3$. Then by #1.9, x solves

$$x' = \left(-\frac{3}{t}\right)x + \frac{3}{t}$$

Thm 1.6

w/ $p(t) = -\frac{3}{t}$

$q(t) = \frac{3}{t}$

$$\Rightarrow e^{\int_{t_0}^{t} p(\tau)\,d\tau} = e^{-3\int_{t_0}^{t} \frac{1}{t}\,d\tau} = e^{-3[\ln(t) - \ln(t_0)]} = \frac{1}{e^{\ln\left(\left(\frac{t}{t_0}\right)^3\right)}} = \left(\frac{t_0}{t}\right)^3$$

and $e^{-\int p(\tau)} = \left(\frac{t}{t_0}\right)^3$

$$\Rightarrow x(t) = \left(\frac{t_0}{t}\right)^3 x_0 + \int_{t_0}^{t} \left(\frac{s}{t_0}\right)^3 \left(\frac{3}{s}\right) ds$$

$$= \left(\frac{t_0}{t}\right)^3 x_0 + \frac{3}{t_0^3} \underbrace{\int_{t_0}^{t} s^2\,ds}_{\left.\frac{s^3}{3}\right|_{t_0}^{t} = \frac{t^3}{3} - \frac{t_0^3}{3}}$$

$$= \left(\frac{t_0}{t}\right)^3 x_0 + \frac{\cancel{3}}{t_0^3}\left[\frac{t^3}{\cancel{3}} - \frac{t_0^3}{\cancel{3}}\right] = \left(\frac{t_0}{t}\right)^3 x_0 + \left(\frac{t}{t_0}\right)^3 - 1$$

Therefore,

$$y(t) = \sqrt[3]{x} = \sqrt[3]{\left(\frac{t_0}{t}\right)^3 x_0 + \left(\frac{t}{t_0}\right)^3 - 1}$$