HW10 MTH 416/616 Spring 2025

Saturday, April 19, 2025 12:48 PM

#5.9 sh soln of $\begin{cases} x''-x=0 \\ x(0)=0,\ x'(0)=1 \end{cases}$ and ch soln of $\begin{cases} x''-x=0 \\ x(0)=1,\ x'(0)=0 \end{cases}$

Show $sh' = ch$

ch solves $\begin{cases} x''-x=0 \\ x(0)=1,\ x'(0)=0 \end{cases}$

$sh''-sh=0 \to sh''=sh$

Let $z = sh' \longrightarrow z' = sh'' = sh$

$\downarrow$

$z'' = sh' = z$

So sh' solves $x''-x=0$.

Now check init conds:

$z(0) = sh'(0) = 1$

$z'(0) = sh(0) = 0$

$\longrightarrow$ $z = sh'$ solves $\begin{cases} x''-x=0 \\ x(0)=1,\ x'(0)=0 \end{cases}$

by uniqueness of solns, $sh' = ch$

Show $ch' = sh$

Basically same

Show $sh(-t) = -sh(t)$

Let $w(t) = sh(-t)$

$w'(t) = -sh'(-t)$

$= -ch(-t)$

$$\boxed{\begin{aligned} &w''(t) = +sh(-t) = w(t) \\ \text{and}\quad &w(0) = sh(0) = 0 \\ &w'(0) = -ch(0) = -1 \end{aligned}}$$

Let $z(t) = -sh(t)$

$z'(t) = -ch(t)$

$$\boxed{\begin{aligned} &z''(t) = -sh(t) = z(t) \\ &z(0) = -sh(0) = 0 \\ &z'(0) = -ch(0) = -1 \end{aligned}}$$

same equation + initial data

$\downarrow$

uniqueness of solns shows that

$sh(-t) = -sh(t)$

Show $ch(-t) = ch(t)$

same argument

Show $ch^2(t) - sh^2(t) = 1$

Our ODE is $x''-x=0$ $\longleftarrow$ of form $Lx=0$ where $p=1, q=-1$

Thus, by Abel thm, the wronskian obeys

$$W[sh, ch] = \frac{C}{1} = C$$

But

$$C = W[sh, ch] = \det\begin{bmatrix} sh & ch \\ sh' & ch' \end{bmatrix} = \det\begin{bmatrix} sh & ch \\ ch & sh \end{bmatrix} = sh^2 - ch^2$$

At $t=0$,

$$C = W[sh, ch](0) = sh^2(0) - ch^2(0) = 0 - 1 = -1$$

Thus $C = -1$

So we have

$$-1 = sh^2 - ch^2, \text{ i.e. } ch^2 - sh^2 = 1$$

#5.10 Find Cauchy function for...

(i) $(e^{-3t}x')' + 2e^{-3t}x = 0$

$p(t)$

$\downarrow$ expand 1st term

$e^{-3t}x'' - 3e^{-3t}x' + 2e^{-3t}x = 0$

$\downarrow$ mult e^{3t}

$x'' - 3x' + 2x = 0$

$\left.\begin{aligned} &r^2 - 3r + 2 = 0 \\ &(r-2)(r-1) = 0 \\ &r = 2, 1 \end{aligned}\right\}$ MTH 335 technique

gen soln

$x(t) = c_1 e^{2t} + c_2 e^{t}$

Now our Cauchy function takes form

$x(t,s) = \alpha(s)e^{2t} + \beta(s)e^{t}$ $\longleftarrow$ def of Cauchy fnct requires $x(s,s) = 0$, $x'(s,s) = \frac{1}{p(s)} = \frac{1}{e^{-3s}} = e^{3s}$

So calculate

$x'(t,s) = 2\alpha(s)e^{2t} + \beta(s)e^{t}$

really it's $\frac{\partial x}{\partial t}$

Now impose the required conditions:

$$\overset{\text{given}}{0 = x(s,s)} = \overset{\text{computed}}{\alpha(s)e^{2s} + \beta(s)e^{s}} \quad (i)$$

$$e^{3s} = x'(s,s) = 2\alpha(s)e^{2s} + \beta(s)e^{s} \quad (ii)$$

From (i) $\longrightarrow$ $\alpha(s) = -\dfrac{\beta(s)e^{s}}{e^{2s}} = -\beta(s)e^{-s}$

$\downarrow$ (ii)

$e^{3s} = 2\left[-\beta(s)e^{-s}\right]e^{2s} + \beta(s)e^{s}$

$= -2\beta(s)e^{s} + \beta(s)e^{s}$

$= -\beta(s)e^{s}$

Thus,

$\beta(s) = -\dfrac{e^{3s}}{e^{s}} = -e^{2s}$ $\longrightarrow$ $\alpha(s) = e^{2s}e^{-s} = e^{s}$

Thus the Cauchy fnct is

$$\boxed{x(t,s) = e^{s}e^{2t} - e^{2s}e^{t}}$$

(ii) $(e^{-10t}x')' + 25e^{-10t}x = 0$

$e^{-10t}x'' - 10e^{-10t}x' + 25e^{-10t}x = 0$

$\downarrow$ mult by e^{10t}

$x'' - 10x' + 25x = 0$

$r^2 - 10r + 25 = 0$

$(r-5)(r-5) = 0$

$r = 5$ (double root)

$\downarrow$

gen soln is

$x(t) = c_1 e^{5t} + c_2 t e^{5t}$

So Cauchy fnct is

$x(t,s) = \alpha(s)e^{5t} + \beta(s)te^{5t}$

$x'(t,s) = 5\alpha(s)e^{5t} + \beta(s)\left[e^{5t} + 5te^{5t}\right]$

$$\overset{\text{by def}}{0} = x(s,s) = \alpha(s)e^{5s} + \beta(s)se^{5s} \quad (i)$$

$$e^{10s} = x'(s,s) = \left[5\alpha(s) + \beta(s)\right]e^{5s} + 5\beta(s)se^{5s} \quad (ii)$$

(i) $\longrightarrow$ $\alpha(s) = -\dfrac{\beta(s)se^{5s}}{e^{5s}} = -s\beta(s)$

$\downarrow$ (ii)

$e^{10s} = \left[-5s\beta(s) + \beta(s)\right]e^{5s} + 5\beta(s)se^{5s}$

$= \beta(s)e^{5s}\left[\cancel{-5s} + 1 + \cancel{5s}\right]$

$\downarrow$

$\beta(s) = \dfrac{e^{10s}}{e^{5s}} = e^{5s}$

$\downarrow$

$\alpha(s) = -se^{5s}$

Thus the Cauchy function is

$$\boxed{x(t,s) = -se^{5s}e^{5t} + e^{5s}te^{5t}}$$

rest of them are similar!