Quiz 7 MTH 335 Fall 2025

Monday, October 13, 2025 11:05 AM

Solve $y'' - y = 2t^2 - t - 3$ call this $f(t)$

Soln: First solve the homogeneous problem

$(*)$ $y'' - y = 0$

Make guess $y(t) = e^{rt}$ and substitute into $(*)$ to get

$r^2 - 1 = 0 \Rightarrow r = \pm 1$

Thus homogeneous solution is

$$y_h(t) = c_1 \underbrace{e^t}_{\text{call this } y_1} + c_2 \underbrace{e^{-t}}_{\text{call this } y_2}$$

Now compute the wronskian

$$W\{y_1, y_2\}(t) = \det\left(\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\right) = e^t(-e^{-t}) - e^t e^{-t} = -2$$

Now we will find the particular solution $y_p(t) = u_1 y_1 + u_2 y_2$

Compute

$$u_1(t) = -\int \frac{y_2(t) f(t)}{W\{y_1, y_2\}(t)}\, dt = -\int \frac{e^{-t}[2t^2 - t - 3]}{(-2)}\, dt$$

$$= \frac{1}{2}\Big[2\underbrace{\int t^2 e^{-t}\, dt - \int t e^{-t} dt - 3\int e^{-t} dt}_{\text{compute with integrations by parts}}\Big]$$

$$= \frac{1}{2}\left\{2\left[-e^{-t}(t^2 + 2t + 2)\right] - \left[-e^{-t}(t+1)\right] + 3e^{-t}\right\}$$

$$= e^{-t}\left\{-t^2 - 2t - 2 + \frac{t}{2} + \frac{1}{2} + \frac{3}{2}\right\}$$

$$= e^{-t}\left\{-t^2 - \frac{3}{2}t\right\}$$

$$u_2(t) = \int \frac{y_1(t) f(t)}{W\{y_1, y_2\}(t)}\, dt = \int \frac{e^t(2t^2 - t - 3)}{-2}\, dt$$

$2 + \frac{1}{2} - \frac{3}{2}$

$$= \frac{-1}{2}\left[2\int t^2 e^t\, dt - \int t e^t\, dt - 3\int e^t\, dt\right]$$

$$= \frac{-1}{2}\left[2e^t(t^2 - 2t + 2) - e^t(t-1) - 3e^t\right]$$

$$= e^t\left[-t^2 + 2t - 2 + \frac{t}{2} - \frac{1}{2} + \frac{3}{2}\right]$$

$$= e^t\left[-t^2 + \frac{5}{2}t - 1\right]$$

Thus the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$= \cancel{e^{-t}}\left[-t^2 - \frac{3}{2}t\right]\cancel{e^t} + \cancel{e^t}\left[-t^2 + \frac{5}{2}t - 1\right]\cancel{e^{-t}}$$

$$= -t^2 - \frac{3}{2}t - t^2 + \frac{5}{2}t - 1$$

$$= -2t^2 + t - 1$$

Therefore the general solution is

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2 e^{-t} - 2t^2 + t - 1}$$