Quiz 11 MTH 335 Fall 2025

Monday, October 13, 2025 12:14 PM

Solve $\begin{cases} y'' + 4y = 2e^{-t} \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

Soln: Recall $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$

Take $\mathcal{L}$ of the ODE to get

$$\Big[s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\Big] + 4\mathcal{L}\{y\} = 2\underbrace{\mathcal{L}\{e^{-t}\}}_{=\frac{1}{s+1}\text{ by the table}}$$

Simplifying gives

$$s^2\mathcal{L}\{y\} - s + 4\mathcal{L}\{y\} = \frac{2}{s+1}$$

$$(s^2+4)\,\mathcal{L}\{y\} = \frac{2}{s+1} + s$$

$$(*) \qquad \mathcal{L}\{y\} = \frac{2}{(s+1)(s^2+4)} + \frac{s}{s^2+4}$$

can invert w/ table

need partial fractions:

$$\frac{2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$\downarrow$ multiply by common denom

$$2 = A(s^2+4) + (Bs+C)(s+1)$$
$$= As^2 + 4A + Bs^2 + Bs + Cs + C$$

$$\Rightarrow \quad 0s^2 + 0s + 2 = (A+B)s^2 + (B+C)s + (4A+C)$$

equate coeffs to get

$$\begin{cases} A+B=0 \to \boxed{A=-B} \\ B+C=0 \to \boxed{B=-C} \\ 4A+C=2 \end{cases} \to A = C$$

$4C + C = 2$

$5C = 2$

$C = \frac{2}{5}$

$B = -2/5$

$A = 2/5$

Thus we have

$$\frac{2}{(s+1)(s^2+4)} = \frac{2/5}{s+1} + \frac{2}{5}\left[\frac{-s+1}{s^2+4}\right] = \frac{2}{5}\left(\frac{1}{s-(-1)}\right) - \frac{2}{5}\left(\frac{s}{s^2+2^2}\right) + \frac{1}{5}\left[\frac{2}{s^2+2^2}\right]$$

So, by table, we see

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+4)}\right\} = \frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)$$

Now returning to $(*)$, we see

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$= \left[\frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t)\right] + \cos(2t)$$

$$= \frac{2}{5}e^{-t} + \frac{3}{5}\cos(2t) + \frac{1}{5}\sin(2t)$$