HW2 MTH 452/552 Spring 2024

p.175 #35 Let $x, y \in F$ meaning $x, y : \mathbb{R} \to \mathbb{R}$

$$\phi_a(x \cdot y) = (xy)(a) = (x(a))(y(a)) = \phi_a(x)\phi_a(y)$$

#38 Show $a^2 - b^2 = (a-b)(a+b)$ $\forall a, b \in R$ iff R commutative

Pf: $(\rightarrow)$ if $a^2 - b^2 = (a-b)(a+b)$

Then

$$a^2 - b^2 = (a-b)a + (a-b)b$$
$$= a^2 - ba + ab - b^2$$

$\downarrow$ subtr a^2 add b^2

$0 = -ba + ab \Rightarrow ab = ba$, thus R commutative

$(\leftarrow)$ if R commutative,

$$a^2 - b^2 = a^2 + ab - ab - b^2$$

$ab = ba$

$$= a^2 + \underbrace{ab}\underbrace{-ba} - b^2$$
$$= a(a+b) - b(a+b)$$
$$= (a-b)(a+b)$$

#44 Let $X = \{a \in \mathbb{R} : a^2 = a\}$ ~~denote~~ the set of idempotent elements. of R. commutative ring

(a) Let $x, y \in X$. Then $x^2 = x$ and $y^2 = y$.

Then $xy \in X$ because $(xy)^2 = (xy)(xy) \underset{R \text{ commutative}}{=} x^2y^2 \overset{x,y \in X}{=} xy$

(b) idempotents of $\mathbb{Z}_6 \times \mathbb{Z}_{12}$:

idempotents of $\mathbb{Z}_6$: 0, 1,

idempotents of $\mathbb{Z}_{12}$: 0, 1, 4, 9, 11

So idempotents in $\mathbb{Z}_6 \times \mathbb{Z}_{12}$: (0,0), (0,4), (0,9), (0,1), (0,11)
(1,0), (1,4), (1,9), (1,1), (1,11)

$2^2 = 4$ x
$3^2 = 9 \bmod 6 = 3$
$4^2 = 16 \bmod 6 = 4$
$5^2 = 25 \bmod 6 = 1 \bmod 6$

n	n^2 mod 12	
2	4	x
3	9	x
4	4	✓
5	1	x
6	0	x
7	1	
8	4	
9	9	
10	4	
11	11	

#1 Soln of $x^3-2x^2-3x=0$ in $\mathbb{Z}_{12}$

x	x^3-2x^2-3x mod 12
0	0
1	8
2	6
3	0
4	8
5	0
6	6
7	8
8	0
9	0
10	2
11	0

$\Rightarrow$ solns are $\{0,3,5,8,9,11\}$

#2 $3x=2$ in $\mathbb{Z}_7$ and $\mathbb{Z}_{23}$

solns are $\{3\}$ — solns are $\{16\}$

(not surprising, both rings are fields so $x=3^{-1}2$ in both)

#3 $x^2+2x+2=0$ in $\mathbb{Z}_6$: Solns $\rightarrow$ NONE!! See:

x	x^2+2x+2 mod 6
0	2
1	5
2	4
3	5
4	2
5	1

#5 characteristic$(2\mathbb{Z})=0$

because $1+1+1+1+\cdots$ never returns to zero

#6 $\text{chr}(\mathbb{Z}\times\mathbb{Z})=0$ since $(1,1)\xrightarrow{\text{never returns to } 0}$

#9 $\text{chr}(\mathbb{Z}_3\times\mathbb{Z}_4)$

$(1,1)\rightarrow(2,2)\rightarrow(0,3)\rightarrow(1,0)\rightarrow(2,1)\rightarrow(0,2)\rightarrow(1,3)\rightarrow(2,0)$

$\downarrow$

$(0,0)\leftarrow(2,3)\leftarrow(1,2)\leftarrow(0,1)$

$\Downarrow$

$\text{chr}(\mathbb{Z}_3\times\mathbb{Z}_4)=12$

#10 $\text{chr}(\mathbb{Z}_6 \times \mathbb{Z}_{15})$

$$\underbrace{(1,1) \to (2,2) \to \cdots \to (0,6)}_{6 \text{ steps}} \to \overbrace{(1,7) \to \cdots \to (0,12)}^{6 \text{ steps}}$$

$$\downarrow$$

$$(0,3) \leftarrow \underbrace{\cdots \leftarrow (3,0) \leftarrow (2,14) \leftarrow (1,13)}_{6 \text{ steps}}$$

$$\downarrow$$

$$\underbrace{(1,4) \xrightarrow{\cdots} (0,9)}_{6 \text{ steps}} \to \underbrace{(1,10) \xrightarrow{\cdots} (0,0)}_{6 \text{ steps}}$$

$\Rightarrow$ 30 steps

$\Rightarrow \text{chr}(\mathbb{Z}_6 \times \mathbb{Z}_{15}) = 30$

#11 $\text{chr}(R) = 4$, R commutative

$\Downarrow$

$$(a+b)^4 = a^4 + \cancel{4a^3b} + 6a^2b^2 + \cancel{4ab^3} + b^4$$

0 since $\text{chr}(R) = 4$

0 since $\text{chr}(R) = 4$

$\cancel{4a^2b^2} + 2a^2b^2$

0 since $\text{chr}(R) = 4$

$$= a^4 + 2a^2b^2 + b^4$$

#12 $\text{chr}(R) = 3$, R commutative

$$(a+b)^9 = a^9 + \cancel{9a^8b} + \cancel{36a^7b^2} + \cancel{84a^6b^3} + \cancel{126a^5b^4} + \cancel{126a^4b^5} + \cancel{84a^3b^6} + \cancel{36a^2b^7} + \cancel{9ab^8} + b^9$$

0   0   0   0   0 since $126 = 3 \cdot 42$

0 since $84 = 3 \cdot 28$ 0   0

$$= a^9 + b^9$$

#24 Show intersection of subdomains of an integral domain D is a subdomain of D.

Pf: Let X_1, X_2 be subdomains of D, i.e. X_1 and X_2 are subrings and are commutative with unity and no zero divisors.

By Exercise 49(a) on p. 170, we know $X_1 \cap X_2$ is a subring of D.

$X_1 \cap X_2$ must be commutative (else there is some $a, b \in X_1 \cap X_2$ s.t. $ab \neq ba$ but that can't happen b/c X_1 commutative)

Since X_1 and X_2 have unity, $X_1 \cap X_2$ also has that same unity.

Thus $X_1 \cap X_2$ is a subring that is commutative and has unity.

Remains to show it contains no zero divisors.

Suppose $a, b \in X_1 \cap X_2$ are zero divisors. Then $a \neq 0$, $b \neq 0$, and $ab = 0$.

But that can't happen b/c it is a calculation in, say, X_1 which has no zero divisors.

Thus $X_1 \cap X_2$ is a subdomain. ∎