HW1 MTH 452/552 Sp 2024

p.142 (#5) $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,1)\rangle$

4 elts

Soln: $\langle(1,1)\rangle = \{(1,1),(0,2),(1,3),(0,4)\}$

But $\mathbb{Z}_2 \times \mathbb{Z}_4$ has 8 elements.

So,

$(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,1)\rangle$ has $4/2 = 2$ elements

(#6) $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$

6 elements

Soln: $\langle(4,3)\rangle = \{(4,3),(8,6),(0,9),(4,12),(8,15),(0,0)\}$

But $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$ has $(12)(18) = 216$ elements

$\Rightarrow (\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle(4,3)\rangle$ has $\frac{216}{6} = 36$ elements

#11 order of $(2,1)+\langle(1,1)\rangle \in (\mathbb{Z}_3\times\mathbb{Z}_6)/\langle(1,1)\rangle$

Soln: $\langle(1,1)\rangle = \{(1,1),(2,2),(0,3),(1,4),(2,5),(0,0)\}$

$$(2,1)+\langle(1,1)\rangle = \{(0,2),(1,3),(2,4),(0,5),(1,0),(2,1)\}$$

$$\downarrow \text{add } (2,1)$$

$$\{(2,3),(0,4),(1,5),(2,0),(0,1),(1,2)\}$$

$$\downarrow \text{add } (2,1)$$

$$\{(1,4),(2,5),(0,0),(1,1),(2,2),(0,3)\}$$

$\Rightarrow$ order of $(2,1)+\langle(1,1)\rangle \in (\mathbb{Z}_3\times\mathbb{Z}_6)/\langle(1,1)\rangle$ is 3

#15 order of $(2,0)+\langle(4,4)\rangle \in (\mathbb{Z}_6\times\mathbb{Z}_8)/\langle(4,4)\rangle$

Soln: $\langle(4,4)\rangle = \{(4,4),(2,0),(0,4),(4,0),(2,4),(0,0)\}$

$(2,0)+\langle(4,4)\rangle = \{(0,4),(4,0),(2,4),(0,0),(4,4),(2,0)\}$ same!

Thus

$$|(2,0)+\langle(4,4)\rangle| = 1$$

#31 Thm: The intersection of normal subgroups is normal.

Pf: Let $N_1, N_2 \subseteq G$ be normal subgroups of G, i.e. by Thm 14.13, for all $g\in G$ and $n_1\in N_1$, $gn_1g^{-1}\in N_1$ and for all $g\in G$ and $n_2\in N_2$, $gn_2g^{-1}\in N_1$.

Let $N=N_1\cap N_2$, and we know $N_1\cap N_2$ is a subgroup of G by a theorem. It remains to show that N is normal. Let $g\in G$ and let $n\in N$. Consider gng^{-1}. We know $n\in N_1$ and $n\in N_2$, so since N_1 and N_2 are normal, $gng^{-1}\in N_1$ and $gng^{-1}\in N_2$.

Thus $gng^{-1}\in N_1\cap N_2 = N$, completing the proof. ∎

p.174

#1 $(12)(16) = 192 \bmod 24 = 0$ ← $192 = (8)(24)$

#3 $(11)(-4) = \underbrace{-44 \bmod 15 =}_{\text{add } 15} \underbrace{-29 \bmod 15 =}_{\text{add } 15} \underbrace{-14 \bmod 15}_{+15} = 1 \bmod 15 = 1$

#5 $(2,3)(3,5) = ((2)(3) \bmod 5,\ (3)(5) \bmod 9)$

$= (6 \bmod 5,\ 15 \bmod 9) = (1,6)$

#6 $(-3,5)(2,-4) = (-6 \bmod 4,\ -20 \bmod 11) = (-2 \bmod 4,\ -9 \bmod 11)$

add $(4,11)$

$= (2 \bmod 4,\ 2 \bmod 11) = (2,2)$

~~#7 ring ✓ commutative ✓ unity ✓~~ field ×

	#7 $n\mathbb{Z}$	#8 $\mathbb{Z}^+$	#9 $\mathbb{Z}\times\mathbb{Z}$	#10 $2\mathbb{Z}\times\mathbb{Z}$
ring	✓	× not group under +	✓	✓
commutative	✓		✓	✓
unity	× no 1 unless n=1		✓	× no 1
field	× no mult. inv.		× no mult inv	×

#14 units of $\mathbb{Z}$: $1, -1$

#15 — $(1,1), (-1,-1), (1,-1), (-1,1)$

~~#16 and no others ($\mathbb{Z}_5$ is~~

#16 — $1, 2, 3, 4$ ($\mathbb{Z}_5$ is a field since 5 prime)

#17 $\mathbb{Q} \setminus \{0\}$ ($\mathbb{Q}$ is a field)

GRAD STUDENTS

P.175 #22 $\det : M_n(\mathbb{R}) \to \mathbb{R}$

Not a ring homomorphism b/c it is not a homomorphism of the addition:

$$\det\left(\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}\right) = 2-4 = -1$$

but

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1 \quad \text{and} \quad \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0$$

shows $\Rightarrow$

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \neq \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$