# HW9 MTH 427/527 Fall 2024

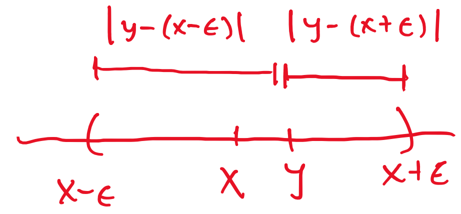
Tuesday, November 5, 2024 11:02 AM

$A^{\circ} = \{x : x \text{ is } \underbrace{\text{an interior pt of } A}_{\exists \epsilon > 0,\ (x-\epsilon, x+\epsilon) \subset A}\}$

**Ex 4.2.1** If $A \subseteq \mathbb{R}$, then $A^{\circ}$ is open.

**Proof**: Let $x \in A^{\circ}$, so $x$ is an interior point of $A$, meaning that $\exists\, \epsilon > 0$ $(x-\epsilon, x+\epsilon) \subset A$. We claim that $(x-\epsilon, x+\epsilon) \subset A^{\circ}$.

Let $y \in (x-\epsilon, x+\epsilon)$ and choose $\epsilon_y = \min\left\{ \underbrace{\frac{|y-(x-\epsilon)|}{2}}_{\text{half distance b/w } y \text{ and } x-\epsilon}, \underbrace{\frac{|y-(x+\epsilon)|}{2}}_{\text{half distance b/w } y \text{ and } x+\epsilon} \right\}$




Then $(y-\epsilon_y, y+\epsilon_y) \subset (x-\epsilon, x+\epsilon) \subset A$. This shows $y \in A^{\circ}$ for arbitrary $y \in (x-\epsilon, x+\epsilon)$.

Thus $(x-\epsilon, x+\epsilon) \subset A^{\circ}$, completing the proof. ■

**Ex 4.2.2** Show $A$ open iff $A = A^{\circ}$

**Proof**: $(\leftarrow)$ If $A = A^{\circ}$, then exercise 4.2.1 shows $A$ is open.

$(\rightarrow)$ Assume $A$ is open, i.e. for all $x \in A$ $\exists\, \epsilon > 0$ $(x-\epsilon, x+\epsilon) \subset A$.

This means all elements of $A$ are interior points, i.e. $A \subset A^{\circ}$.

Let $y \in A^{\circ}$, then $\exists\, \epsilon_y > 0$ so that $(y-\epsilon_y, y+\epsilon_y) \subset A$, hence $y \in A$.

Thus $A^{\circ} \subset A$ and so we have shown $A = A^{\circ}$, completing the proof. ■

**Ex 4.2.3** Let $\mathcal{U} \subseteq \mathbb{R}$ be nonempty open set. Show $\sup \mathcal{U} \notin \mathcal{U}$ and $\inf \mathcal{U} \notin \mathcal{U}$.

**Proof**: Suppose $\sup \mathcal{U} \in \mathcal{U}$, and let $\epsilon > 0$. Since $\mathcal{U}$ open, $\exists\, \epsilon_{\mathcal{U}} > 0$ so that $(\sup(\mathcal{U}) - \epsilon_{\mathcal{U}}, \sup(\mathcal{U}) + \epsilon_{\mathcal{U}}) \subset \mathcal{U}$.

**Claim**: $(\sup(\mathcal{U}) - \epsilon, \sup(\mathcal{U}) + \epsilon) \cap (\mathbb{R} \setminus \mathcal{U})$ is nonempty

**Proof of claim**: Suppose it is empty. This means $(\sup \mathcal{U} - \epsilon, \sup \mathcal{U} + \epsilon) \subset \mathcal{U}$.

But that means $\sup(\mathcal{U}) + \frac{\epsilon}{2} \in \mathcal{U}$. This can't be the case because $\sup(\mathcal{U})$ is the least upper bound of $\mathcal{U}$.

Thus it can't be the case that $(\sup(\mathcal{U}) - \epsilon, \sup(\mathcal{U}) + \epsilon)$ is empty.

Since for all $\epsilon > 0$, $(\sup(\mathcal{U}) - \epsilon, \sup(\mathcal{U}) + \epsilon)$ is not a subset of $\mathcal{U}$, a contradiction.

Thus $\sup \mathcal{U} \notin \mathcal{U}$. ■

The proof that $\inf \mathcal{U} \notin \mathcal{U}$ is similar.