HW8 MTH427/527 Fall 2024

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Exercises 2.2.7, 2.2.9, 2.2.10

Exercise 2.2.7. Suppose $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ are infinite series for which $a_i \geq 0$ and $b_i > 0$ for all $i \geq m$. Show that if $\sum_{i=m}^{\infty} b_i$ converges and

partial sums to converge to some T

$$\lim_{i\to\infty} \frac{a_i}{b_i} < +\infty, \tag{2.2.14}$$

partial sums S_n converge to some S

then $\sum_{i=m}^{\infty} a_i$ converges.

Proof: Suppose $a_i \geq 0$, $b_i > 0$, that $\sum_{i=m}^{\infty} b_i =: T$ converges, and that $\lim_{i\to\infty} \frac{a_i}{b_i} =: c < \infty$.

Let $\epsilon > 0$. Since $\sum_{i=m}^{\infty} b_i = T$, $\exists N_1 \forall n > N_1$ that partial sums $t_n = \sum_{i=m}^{n} b_i$ obey $|t_n - T| < \epsilon$.

Since $\lim_{i\to\infty} \frac{a_i}{b_i} = c$, $\exists N_2 \forall n > N_2$, $\left|\frac{a_n}{b_n} - c\right| < \epsilon$, i.e. $-\epsilon < \frac{a_n}{b_n} - c < \epsilon$, i.e.

$c - \epsilon < \frac{a_n}{b_n} < c + \epsilon$, i.e.

$b_n(c-\epsilon) < a_n < b_n(c+\epsilon)$

In particular, we see $a_n < b_n(c+\epsilon)$.

Therefore by Prop 2.2.6 we conclude that $\sum_{i=m}^{\infty} a_i$ converges. ∎

Exercise 2.2.9. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

converges.

Exercise 2.2.10. Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for any real number $x \geq 0$.

2.2.9: Take $a_n = \frac{1}{n2^n}$ and $b_n = \frac{1}{2^n}$. Then $\sum_{n=1}^{\infty} b_n$ converges because it is a geometric series. Also,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ by Ex 2.1.16}$$

Therefore by Exercise 2.2.7, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges. ∎

2.2.10 | First assume $0 < x < 1$, so $\sum_{n=0}^{\infty} x^n$ converges by Proposition 2.2.2.

So let $a_n = \frac{x^n}{n!}$ and $b_n = x^n$ and we have

(#1) $\sum b_n$ converges and

(#2) $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{x^n}{n!}}{x^n} = \lim_{n\to\infty} \frac{1}{n!} \overset{\text{Ex 2.2.1 and Prop 2.2.1}}{=} 0$

Therefore, by Exercise 2.2.7, $\sum_{n=0}^{\infty} a_i = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges in this $0 < x < 1$ case.

Now suppose $x \geq 1$. Now we no longer have $\sum x^n$ converging. So instead first consider $x = 1$. This case is proved by Example 2.2.1. So finally assume $x > 1$.

Choose now $b_n = \frac{1}{x^n}$, because $x > 1 \Rightarrow \frac{1}{x} < 1$.

So we see that $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$ converges since it is a geometric series.

Let $a_i = \frac{x^n}{n!}$. It remains to argue that $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$.

So, compute

$$\frac{a_n}{b_n} = \frac{\frac{x^n}{n!}}{\frac{1}{x^n}} \leq \frac{x^{2n}}{n!} = \frac{(x^2)^n}{n!}$$

We see that for all n, $x^n > x^{n-1} > x^{n-2} > \cdots > x^2 > x > 1$. Let $\epsilon > 0$.

Let N_1 be so large that $N_1 > x^2$. Clearly $\forall n > N_1$, $n^n > x^{2n}$ and $n! < n^n \to \frac{1}{n!} < \frac{1}{n^n}$.

$$\frac{(x^2)^n}{n!} < \frac{n^n}{n^n} < 1$$

$x^{2n-2} < (n-1)^{n-1}$ $\qquad$ $\frac{1}{(n-1)!} < \frac{1}{(n-1)^{n-1}}$

Also choose N_2 s.t. for all $n > N_2$, $\frac{1}{n} < \frac{\epsilon}{x^2}$.

Choose $N = \max\{N_1, N_2\}$. Then $\forall n > N$,

$$\left|\frac{x^{2n}}{n!}\right| = \frac{(x^2)^n}{n!} < \frac{x^2}{n} \frac{(x^2)^{n-1}}{(n-1)!} < \frac{x^2}{n} \frac{(n-1)^{n-1}}{(n-1)^{n-1}} = \frac{x^2}{n} < \left(\frac{\epsilon}{x^2}\right)x^2 = \epsilon.$$

Thus $\lim_{n\to\infty} \frac{x^{2n}}{n!} = 0 < \infty$.

So we have both $\sum b_n = \sum \frac{1}{x^n}$ converges and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0 < \infty$, hence by Exercise 2.2.7,

$\sum a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, completing the proof. ∎