HW7 MTH 427/527 Fall 2024

Monday, October 7, 2024 3:39 PM

Def 2.1.3: $\{a_i\}$ bounded means $\exists M \in \mathbb{R}\ \forall i\ |a_i| \le M$

so $\{a_i\}$ not bounded means $\forall M \in \mathbb{R}\ \exists i\ |a_i| > M$

Exercise 2.1.25 Suppose $\{x_i\}_{i=m}^{\infty}$ is not bounded. Show that either $-\infty$ or ∞ is a subsequential limit of $\{x_i\}_{i=m}^{\infty}$.

Proof: Suppose $\{x_i\}_{i=m}^{\infty}$ is not bounded, i.e. that $\forall M \in \mathbb{R}\ \exists i_M\ |x_{i_M}| > M$.

Now consider these values i_M for $M = 1, 2, 3, \dots$.

Form two sets

$$A = \{x_{i_M} : x_{i_M} > 0\}_{M=1,2,3,\dots} \quad \text{and} \quad B = \{x_{i_M} : x_{i_M} < 0,\ M = 1,2,3,\dots\}$$

Now one of these sets must have infinitely many elements (both could!).

That set then gives a subsequence of $\{x_i\}$ whose limit is either $+\infty$ (if A has ∞-many elts) or $-\infty$ (if B has ∞-many elts). ∎

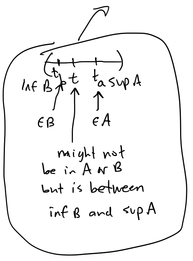
Exercise 2.1.26 Spz $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ where for all $a \in A, b \in B$ $a \le b$. Show that $\sup A \le \inf B$.

Proof: Assume not, i.e. that $\sup A > \inf B$. So $\sup A - \inf B > 0$. Choose $t \in (\sup A, \inf B)$. Then t is NOT an upper bound of A and t is NOT a lower bound of B, so $\exists t_a \in A, t_b \in B$ so that $t_a \ge t$ and $t_b \le t$.

Thus,

$$t_b \le t \le t_a,$$

but that is a contradiction to the hypotheses, completing the proof. ∎



Exercise 2.2.1 Suppose $\sum_{i=m}^{\infty} a_i$ converges and $\beta \in \mathbb{R}$. Show $\sum_{i=m}^{\infty} \beta a_i$ converges and $\sum_{i=m}^{\infty} \beta a_i = \beta \sum_{i=m}^{\infty} a_i$

Proof: Since $\sum_{i=m}^{\infty} a_i$ converges, it means that the partial sums $s_n = \sum_{i=m}^{\infty} a_i$ converge, say $s_n \to S$, so $\forall \epsilon > 0\ \exists N\ \forall n > N\ |s_n - S| < \epsilon$. Let $\epsilon > 0$.

Case 1: $\beta = 0$

If $\beta = 0$, then partial sums $w_n = \sum_{i=m}^{\infty} \beta a_i = \sum_{i=m}^{\infty} 0 = 0$ and we see $w_n \to 0$, i.e. that $\sum_{i=m}^{\infty} \beta a_i = 0$. But also $\beta \sum_{i=m}^{\infty} a_i = 0 \cdot S = 0$, completing the proof in this case.

Case 2: $\beta \ne 0$

If $\beta \ne 0$, choose N so that $\forall n > N$ $|s_n - S| < \frac{\epsilon}{|\beta|}$. Then define the partial sums $w_n = \sum_{i=m}^{n} \beta a_i = \beta s_n$. So compute

$$|w_n - \beta S| = |\beta s_n - \beta S| = |\beta|\,|s_n - S| < |\beta| \frac{\epsilon}{\beta} = \epsilon,$$

completing the proof. ∎

Exercise 2.2.2 Suppose both $\sum_{i=m}^{\infty} a_i$ and $\sum_{i=m}^{\infty} b_i$ converge. Show that $\sum_{i=m}^{\infty} (a_i + b_i)$ converges and $\sum_{i=m}^{\infty} (a_i + b_i) = \sum_{i=m}^{\infty} a_i + \sum_{i=m}^{\infty} b_i$.

Proof: Let $S = \sum_{i=m}^{\infty}$ and $T = \sum_{i=m}^{\infty} b_i$ and let $s_n = \sum_{i=m}^{n} a_i$ and $t_n = \sum_{i=m}^{n} b_i$ be the partial sums, so we know $s_n \to S$ and $t_n \to T$. Let $w_n = \sum_{i=m}^{n} (a_i + b_i) = s_n + t_n$.

Let $\epsilon > 0$ and choose N_1 so that $\forall n > N_1$, $|s_n - S| < \frac{\epsilon}{2}$. Choose N_2 so that $\forall n > N_2$, $|t_n - T| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then,

$$\begin{aligned} |w_n - (S + T)| &= |s_n + t_n - S - T| \\ &= |(s_n - S) + (t_n - T)| \\ &\overset{\Delta\text{-ineq}}{\le} |s_n - S| + |t_n - T| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

completing the proof. ∎

Exercise 2.2.5 Show that for $x \ne 1$ that

$$(\star) \qquad s_n := \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$

Proof: For $n = 0$, the left-hand side of $(\star)$ becomes

$$s_0 := \sum_{i=0}^{0} x^i = x^0 = 1$$

and the right-hand side of $(\star)$ becomes

$$\frac{1 - x^{0+1}}{1 - x} = \frac{1 - x}{1 - x} = 1$$

So the equality $(\star)$ holds for $n = 0$.

Now assume $(\star)$ holds for $n = N$, i.e. that $s_N := \sum_{i=0}^{N} x^i = \frac{1 - x^{N+1}}{1 - x}$. Induction hypothesis

Now compute

$$\begin{aligned} s_{N+1} := \sum_{i=0}^{N+1} x^i &\overset{\text{remove last term of sum}}{=} \left(\sum_{i=0}^{N} x^i\right) + x^{N+1} = \frac{1 - x^{N+1}}{1 - x} + x^{N+1} \\ &= \frac{1 - x^{N+1} + x^{N+1}(1 - x)}{1 - x} \\ &= \frac{1 - \cancel{x^{N+1}} + \cancel{x^{N+1}} - x^{N+2}}{1 - x} \\ &= \frac{1 - x^{N+2}}{1 - x}, \end{aligned}$$

which is $(\star)$ with $n = N+1$, completing the proof. ∎