HW6 MTH 427/527 Fall 2024

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Exercise 2.1.12 Suppose $A \subseteq \mathbb{R}$, $A \neq \emptyset$ and $s = \sup A$. Show there exists a sequence $\{a_i\}$ so that $a_i \in A$ and $\lim_{i\to\infty} a_i = s$.

Proof: Suppose $s = \infty$. That means A does not have an upper bound, as defined on p. 23 in the book. So, for each n, $A \cap [n, \infty)$ is nonempty. So pick each $a_i \in A \cap [i, \infty)$, meaning $a_i > i$. Therefore this sequence meets Definition 2.1.4, completing the proof that $\lim_{i\to\infty} a_i = \infty$ in this case.

Now suppose $s \in \mathbb{R}$. Then s is the least upper bound of A, meaning for any $t < s$, $[t, s] \cap A$ is nonempty. Define a sequence a_n by choosing $a_n = \inf\left(\left[s - \frac{1}{n}, s\right] \cap A\right)$.

Then, we know for each n, $s - \frac{1}{n} \leq a_n \leq s$, so $-\frac{1}{n} \leq a_n - s \leq 0 \leq \frac{1}{n}$, i.e. $|a_n - s| < \frac{1}{n}$.

"obvious"

Choose $N > \frac{1}{\epsilon}$ so that if $n > N > \frac{1}{\epsilon}$, we have $\frac{1}{n} < \epsilon$. Then, for such n,

$$|a_n - s| < \frac{1}{n} < \epsilon,$$

completing the proof. ∎

Exercise 2.1.14 Prove that if $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} y_n = M$, then $\lim_{n\to\infty} x_n + y_n = L + M$.

Proof: Let $\epsilon > 0$. Choose N_x so that for all $n > N_x$, $|x_n - L| < \frac{\epsilon}{2}$. and choose N_y so that for all $n > N_y$, $|y_n - M| < \frac{\epsilon}{2}$. Let $N = \max\{N_x, N_y\}$. Then for all $n > N$, compute

$$|(x_n + y_n) - (L + M)| = |(x_n - L) + (y_n - M)|$$

$$\overset{\Delta\text{-ineq}}{\leq} |x_n - L| + |y_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

completing the proof. ∎

Exercise 2.1.15 Prove that if $\{x_i\}$ and $\{y_i\}$ are convergent sequences with $x_i \to L$ and $y_i \to M$, then $x_i y_i \to LM$.

Proof: Since convergent sequences are bounded, let $B > 0$ be an upper bound of the set $\{|x_i| : i \in I\} \cup \{|y_i| : i \in I\}$, i.e. that for all i, $|x_i| \leq B$ and $|y_i| \leq B$.

First suppose $M = 0$. Then $\forall \epsilon > 0 \, \exists N \, \forall n > N$ $|y_n - 0| = |y_n| < \frac{\epsilon}{B}$

$$|x_i y_i - \overset{0}{\cancel{L}}M| = |x_i y_i| = |x_i||y_i| < B\left(\frac{\epsilon}{B}\right) = \epsilon,$$

completing the proof in this case.

Now suppose $M \neq 0$. Choose N_1 so that $\forall n > N_1$, $|x_i - L| < \frac{\epsilon}{2|M|}$ and choose N_2 so that $\forall n > N_2$, $|y_i - L| < \frac{\epsilon}{2B}$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Then,

$$|x_i y_i - LM| = |x_i y_i \pm x_i M - LM|$$

$$= |x_i||y_i - M| + |M||x_i - L|$$

$$< \frac{B\epsilon}{2B} + \frac{|M|\epsilon}{2|M|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

completing the proof. ∎

Exercise 2.1.16 (a) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof: Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. Then, for $n > N$, $n > \frac{1}{\epsilon}$, hence $\frac{1}{n} < \epsilon$. Thus,

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon,$$ completing the proof. ∎

(b) Show $\lim_{n\to\infty} \frac{1}{n^2}$ directly

Proof: Let $\epsilon > 0$ and choose $N > \frac{1}{\epsilon}$. Then for all $n > N$, $n > \frac{1}{\sqrt{\epsilon}}$, hence $\frac{1}{n} < \sqrt{\epsilon}$. But then,

$$\frac{1}{n^2} < \frac{\sqrt{\epsilon}}{n} = \sqrt{\epsilon} \cdot \left(\frac{1}{n}\right) < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon,$$

completing the proof. ∎

and using previous results

Proof: Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then

$$\lim_{n\to\infty} \frac{1}{n^2} \overset{\text{Prop 2.1.7}}{=} \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) = 0 \cdot 0 = 0$$ ∎

Exercise 2.1.17 Show for $k \in \{1, 2, 3, \ldots\}$, $\lim_{n\to\infty} \frac{1}{n^k} = 0$.

Proof: Repeated application of Prop 2.1.7 and Exercise 2.1.16 shows

$$\lim_{n\to\infty} \frac{1}{n^k} = \underbrace{\left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) \cdots \left(\lim_{n\to\infty} \frac{1}{n}\right)}_{k \text{ factors}}$$

$$= \left(\lim_{n\to\infty} \frac{1}{n}\right)^k = 0^k = 0,$$

completing the proof. ∎

Exercise 2.1.18

$$\lim_{n\to\infty} \frac{3n^5 + 8n^3 - 6n}{8n^5 + 2n^4 - 31} = \lim_{n\to\infty} \left(\frac{3n^5 + 8n^3 - 6n}{8n^5 + 2n^4 - 31}\right)\left(\frac{\frac{1}{n^5}}{\frac{1}{n^5}}\right)$$

$$= \lim_{n\to\infty} \frac{3 + 8/n^2 - 6/n^4}{8 + 2/n^4 - 31/n^5}$$

$$\overset{\text{Prop 2.1.8}}{=} \frac{\lim_{n\to\infty}\left(3 + \frac{8}{n^2} - \frac{6}{n^4}\right)}{\lim_{n\to\infty}\left(8 + \frac{2}{n^4} - \frac{31}{n^5}\right)} \overset{\text{Props 2.15 and 2.16}}{=} \frac{\overset{3 \text{ by Quiz 10}}{\lim_{n\to\infty} 3} + 8 \overset{0}{\lim_{n\to\infty} \frac{1}{n^2}} - 6 \overset{0}{\lim_{n\to\infty} \frac{1}{n^4}}}{\underset{8 \text{ by Quiz 10}}{\lim_{n\to\infty} 8} + 2 \underset{0}{\lim_{n\to\infty} \frac{1}{n^4}} - 31 \underset{0}{\lim_{n\to\infty} \frac{1}{n^5}}}$$

by Ex 2.1.17

$$= \frac{3 + 0 + 0}{8 + 0 + 0} = \frac{3}{8}$$

Exercise 2.1.19

for numerator, $\frac{1}{n} = \frac{1}{\sqrt{n^2}}$

$$\lim_{n\to\infty} \frac{\sqrt{3n^2 + 1}}{5n + 6} = \lim_{n\to\infty} \frac{\sqrt{3n^2 + 1}}{5n + 6}\left(\frac{1/n}{1/n}\right)$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{3n^2 + 1}{n^2}}}{\frac{5n + 6}{n}} = \lim_{n\to\infty} \frac{\sqrt{3 + \frac{1}{n^2}}}{5 + \frac{6}{n}}$$

$$\overset{\text{Prop 2.1.8}}{=} \frac{\lim_{n\to\infty} \sqrt{3 + \frac{1}{n^2}}}{\lim_{n\to\infty} 5 + \frac{6}{n}}$$

$$\overset{\text{Props 2.1.9, 2.1.5, and 2.1.6}}{=} \frac{\sqrt{3}}{5}$$

Exercise 2.1.21 Show that

$\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$

Proof: Suppose that $\lim_{n\to\infty} |a_n| = 0$, i.e. that $\forall \epsilon > 0 \, \exists N \, \forall n > N$ $\big||a_n| - 0\big| = |a_n| < \epsilon$.

Let $\epsilon > 0$. Choose N s.t. $\forall n > N$, $\big||a_n| - 0\big| = \big||a_n|\big| = |a_n| < \epsilon$.

Then,

$$|a_n - 0| = |a_n| < \epsilon,$$

completing the proof.

Conversely, suppose that $\lim_{n\to\infty} a_n = 0$. Let $\epsilon > 0$ and choose N s.t. $\forall n > N$ $|a_n - 0| = |a_n| < \epsilon$.

Then compute

$$\big||a_n| - 0\big| = |a_n| < \epsilon,$$

completing the proof. ∎

Exercise 2.1.22 Suppose $\{x_k\}$ converges with $x_k \to L$. Show every subsequence $\{x_{n_k}\}$ of $\{x_k\}$ also converges with $x_{n_k} \to L$.

Proof: Since $\{x_{n_k}\}$ is a subsequence of $\{x_k\}$, we have $n_k \geq k$. Let $\epsilon > 0$ and choose N so that for all $k > N$, $|x_k - L| < \epsilon$. Then $n_k \geq k > N$, so we see $|x_{n_k} - L| < \epsilon$, completing the proof. ∎

Ex 2.1.24 Suppose $\{x_k\}$ diverges to $-\infty$. Show each subsequence $\{x_{n_k}\}$ also diverges to $-\infty$.

Proof: Since $x_k \to -\infty$, Def 2.1.4 tells us that $\forall M \in \mathbb{R} \, \exists N \, \forall k > N$ $x_k < M$. Let $M \in \mathbb{R}$ and choose N so that $\forall k > N$ $x_k < M$. Since $n_k \geq k$, we have $n_k \geq k > N$, so we also get $x_{n_k} < M$, completing the proof. ∎