HW5 MTH427/527 Fall 2024

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Exercise 2.1.6(c) Find $\limsup_{n\to\infty} 2^{-n}$ and $\liminf_{n\to\infty} 2^{-n}$.

Solution: Compute

$$u_i = \sup\{2^{-j} : j \geq i\} = \sup\{2^{-i}, 2^{-i-1}, 2^{-i-2}, \ldots\} = 2^{-i}$$

and

$$l_i = \inf\{2^{-j} : j \geq i\} = \inf\{2^{-i}, 2^{-i-1}, 2^{-i-2}, \ldots\} = 0$$

Now by Definition 2.1.5,

$$\limsup_{n\to\infty} 2^{-n} = \inf\{u_1, u_2, u_3, \ldots\} = \inf\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots\} = 0$$

and

$$\liminf_{n\to\infty} 2^{-n} = \sup\{l_1, l_2, \ldots\} = \sup\{0\} = 0$$

Exercise 2.1.7 Suppose $\{a_i\}$ and $\{c_k\}$ are sequences so that $\exists N \forall i > N$, $a_i \leq c_i$.

Show that if $\lim_{i\to\infty} a_i = \infty$, then $\lim_{i\to\infty} c_i = \infty$.

Proof: Suppose that $a_i \leq c_i$ and $\lim_{i\to\infty} a_i = \infty$. By Definition 2.1.4, we see that $\forall M \in \mathbb{R}\ \exists N \forall i > N\ a_i > M$. We need to show that $\forall M \in \mathbb{R}\ \exists N \forall i > N\ c_i > M$.

Let $M \in \mathbb{R}$. Since $\lim_{i\to\infty} a_i = \infty$ we know $\exists N \forall i > N\ a_i > M$. For $i > N$, we have

$$c_i \geq a_i > M,$$

completing the proof. ∎

Exercise 2.1.10 Show for any sequence $\{a_i\}$,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Proof: First define

$$l_i = \inf\{a_j : j \geq i\} \quad \text{and} \quad u_i = \sup\{a_j : j \geq i\}.$$

Clearly, $l_i \leq u_i$. Exercise 2.1.5 shows that

$$\liminf_{i\to\infty} a_i = \lim_{i\to\infty} l_i \quad \text{and} \quad \limsup_{i\to\infty} a_i = \lim_{i\to\infty} u_i$$

By Exercise 2.1.9, we see that

$$\liminf_{i\to\infty} a_i = \lim_{i\to\infty} l_i \leq \lim_{i\to\infty} u_i = \limsup_{i\to\infty} a_i,$$

completing the proof. ∎

Exercise 2.1.11 Suppose $u \in \mathbb{R}$ with $u \geq 0$ and $\forall \epsilon > 0,\ u < \epsilon$. Prove that $u = 0$.

Proof: Suppose $u \neq 0$. Then $u > 0$. But choosing $\epsilon = \frac{u}{2}$ implies $u < \frac{u}{2}$, but that is false, so we have a contradiction. Therefore $u = 0$. ∎