HW4 MTH 427/527 Fall 2024

Wednesday, October 2, 2024 1:40 PM

Exercise 2.1.1

Show a nonincreasing, bounded sequence must converge.

Proof: Let a_n be a nonincreasing bounded sequence. This means that for all n,

$a_{n+1} \le a_n$.

Let $L = \inf\{a_i : i \in I\}$, which must exist since the sequence is bounded.

Let $\epsilon > 0$. There must be some N so that $a_N < L + \epsilon$, else L would not be the infimum. So, for all $n > N$,

$$L - \epsilon < L \le a_n \le a_N < L + \epsilon$$

"obvious"

Therefore,

$$-\epsilon < a_n - L < \epsilon,$$

which is equivalent to $|a_n - L| < \epsilon$, which completes the proof. ∎

Def 2.1.4

$\forall M \exists N \forall j > N \; a_j > M$

Exercise 2.1.2

Show that a nondecreasing sequence either converges or diverges to $+\infty$.

Proof: Let a_n be a nondecreasing sequence, so $\forall n$, $a_{n+1} \ge a_n$.

If a_n is a bounded sequence, then Exercise 2.1.1 shows that a_n converges.

So assume that a_n is not bounded, i.e. $\forall M \in \mathbb{R} \, \exists i \; |a_i| > M$.

"a_n bounded" is defined as $\exists M \forall i \; |a_i| \le M$ ← Def 2.1.1

So, "a_n not bounded" means $\forall M \exists i \; |a_i| > M$

We need to show $\forall M \exists N \forall i > N \; a_i > M$.

Let $M \in \mathbb{R}$. Since a_n is not bounded, we know $\exists i$ so that $|a_i| > M$.

Since a_n is nondecreasing, for any $j > i$, $a_j \ge a_i$.

First suppose $a_i \ge 0$. Then

$$a_j \ge a_i = |a_i| > M,$$

completing the proof in this case.

$-a_j \le -a_i$

Now suppose $a_i < 0$. So, $|a_i| = -a_i > M$, hence $a_i < -M$. Then, $-a_j \le -a_i < -M$, hence $a_j > M$, completing the proof. ∎