HW3 MTH427/527 Fall 2024

Sunday, September 8, 2024 6:11 PM

1.3.10

1.3.11

1.4.1

$\lim \frac{n+3}{n+2} = 1$

#1.3.10 Prove: $\nexists\, s \in \mathbb{Q}$ such that $s^2 = 3$.

Proof: Suppose such an $s \in \mathbb{Q}$ existed, i.e. $\exists p, q \in \mathbb{Z}, q \neq 0$ so that $s = \frac{p}{q}$ (written in lowest terms so p and q have no common factor) and $s^2 = 3$.

Hence $\frac{p^2}{q^2} = 3$, thus $\boxed{p^2 = 3q^2}$ $(\star)$. This means p is divisible by 3, so $\exists k \in \mathbb{Z}$ such that $p = 3k$. Substitute this into $(\star)$ to obtain $9k^2 = 3q^2$ and simplify to $3k^2 = q^2$. This implies 3 divides q, but that's a contradiction to the assumption that p and q had no common factor, completing the proof. ∎

#1.4.1 Suppose $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are sequences in $\mathbb{Q}$ with

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i.$$

Show that $a_i \sim b_i$.

$a_i \sim b_i$ means $\forall \epsilon > 0\, \exists N\, \forall n > N\, |a_n - b_n| < \epsilon$

$\lim_{i \to \infty} a_i = L$ means $\forall \epsilon > 0\, \exists N\, \forall n > N\, |a_i - L| < \epsilon$

Proof: Since $\lim_{i \to \infty} a_i$ and $\lim_{i \to \infty} b_i$ exist and are equal, call their limit L. Let $\epsilon > 0$. We need to prove that $\exists N$ so that $\forall n > N$, $|a_n - b_n| < \epsilon$.

Since $\lim_{i \to \infty} a_i = L$, we know $\exists N_a$ such that $\forall n > N_a$, $|a_n - L| < \frac{\epsilon}{2}$.

Similarly since $\lim_{i \to \infty} b_i = L$, $\exists N_b\, \forall n > N_b$, $|b_n - L| < \frac{\epsilon}{2}$.

Let $N = \max\{N_a, N_b\}$ and let $n > N$. Then compute

$$|a_i - b_i| = |a_i - b_i \pm L| = |(a_i - L) + (L - b_i)| \leq |a_i - L| + |b_i - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

"add convenient form of zero" $\quad$ Δ-inequality

Completing the proof. ∎

Prove $\lim_{n \to \infty} \frac{n+3}{n+2} = 1$.

Proof: Let $\epsilon > 0$ and choose $N > \frac{1}{\epsilon} - 2$.

Let $n > N$. Then $n > N > \frac{1}{\epsilon} - 2$.

We see $n + 2 > \frac{1}{\epsilon}$ and so $\frac{1}{n+2} < \frac{1}{1/\epsilon} = \epsilon$.

Finally, compute

$$\left|\frac{n+3}{n+2} - 1\right| = \left|\frac{\cancel{n}+3-\cancel{n}-2}{n+2}\right| = \frac{1}{n+2} < \epsilon,$$

Completing the proof. ∎

"scratch work"

thinking: ultimately want

$\left|\frac{n+3}{n+2} - 1\right| < \epsilon$

$\parallel$

$\left|\frac{\cancel{n}+3-\cancel{n}-2}{n+2}\right|$

$\parallel$

$\frac{1}{n+2}$

$\iff \frac{1}{n+2} < \epsilon$

$\Updownarrow$

$n + 2 > \frac{1}{\epsilon}$

$n > \frac{1}{\epsilon} - 2$