HW2 MTH427/527 Fall 2024

Sunday, September 8, 2024 6:31 PM

1.3.3
1.3.4 (b–d)
1.3.6
1.3.8

1.3.3 Show if $a,b \in \mathbb{Q}^+$, then $a+b \in \mathbb{Q}^+$.

Proof: Let $a,b \in \mathbb{Q}^+$, so $\exists\, p,q,r,s \in \mathbb{Z}^+$ so that $a = \frac{p}{q}$ and $b = \frac{r}{s}$.

Now compute

$$a+b = \frac{p}{q} + \frac{r}{s} \underset{(1.3.6)}{=} \frac{ps+rq}{qs}$$

But since $p,q,r,s \in \mathbb{Z}^+$, $ps+rq \in \mathbb{Z}^+$ and $qs \in \mathbb{Z}^+$.
Thus choosing $x = ps+rq$ and $y = qs$ expresses $\frac{p}{q}+\frac{r}{s}$ as $\frac{x}{y}$ for $x,y \in \mathbb{Z}^+$,
which means $a+b = \frac{p}{q}+\frac{r}{s} \in \mathbb{Q}^+$.

1.3.4 (b) Show "if $a<b$ and $b<c$, then $a<c$"

Proof: Suppose $a<b$ and $b<c$. Then $b-a \in \mathbb{Q}^+$ and $c-b \in \mathbb{Q}^+$.
We need to show $a<c$, i.e. that $c-a \in \mathbb{Q}^+$.

By exercise 1.3.3,

$$(b-a)+(c-b) \in \mathbb{Q}^+$$

but

$$(\cancel{b}-a)+(c-\cancel{b}) = c-a \in \mathbb{Q}^+,$$

as was to be shown. ∎

(c) Show "if $a<b$, then $a+c<b+c$"

Proof: Suppose $a<b$, i.e. that $b-a \in \mathbb{Q}^+$.

$$\text{Then } b-a = b-a \pm c = b+c-a-c$$
$$= (b+c)-(a+c) \in \mathbb{Q}^+,$$

but that is precisely the meaning of $a+c<b+c$, completing the proof. ∎

(d) Show "if $a>0$ and $b>0$, then $ab>0$"

Proof: Suppose $a>0$ and $b>0$, so there exist $p,q,r,s \in \mathbb{Z}^+$ so that $a=\frac{p}{q}$ and $b=\frac{r}{s}$.

Then compute

$$ab = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right) \underset{(1.3.7)}{=} \frac{pr}{qs}$$

But $pr \in \mathbb{Z}^+$ and $qs \in \mathbb{Z}^+$, so writing $x=pr$ and $y=qs$ expresses

$$ab = \frac{x}{y} \text{ with } x,y \in \mathbb{Z}^+,$$

which is what $ab>0$ means, completing the proof. ∎

1.3.8 Show "$\forall a \in \mathbb{Q}, -|a| \le a \le |a|$"

Proof: First suppose $a \ge 0$. Then $|a| = a$ and so $a \le a = |a|$ and $-|a| = -a \le a$.
Thus $-|a| \le a \le |a|$ in this case.
Now suppose $a<0$. Then $|a| = -a$ and so $-|a| = -(-a) = a \le a$
and $a = -|a| \le |a|$.
Thus $-|a| \le a \le |a|$, completing the proof. ∎