MTH 427/527 Fall 2024

1.1.3 Assume R is an equivalence relation, so R is symmetric, reflexive, and transitive.

(a) Prove $[x] \cap [y] \neq \emptyset$ iff $x \sim_R y$

Proof: ($\rightarrow$) Suppose that $[x] \cap [y] \neq \emptyset$. Thus there is some $w \in [x] \cap [y]$. Since $w \in [x]$, $w \sim_R x$ and since $w \in [y]$, $w \sim_R y$. Since R symmetric, we know from $w \sim_R x$ that $x \sim_R w$. Now we have $x \sim_R w$ and $w \sim_R y$, so by transitive property, we have $x \sim_R y$.

($\leftarrow$) Suppose $x \sim_R y$. Since R is reflexive, we know that $x \in [x]$ and $y \in [y]$. Since $x \sim_R y$, we conclude also that $y \sim_R x$ by R being symmetric. Thus we see that both $x \in [y]$ and $y \in [x]$. This means that $\{x, y\} \subseteq [x] \cap [y]$, so $[x] \cap [y] \neq \emptyset$.

We proved both directions of the iff, so the proof is complete. ∎

(b) graded

1.1.4 $[1] = \{\ldots, -3, -1, 1, 3, \ldots\}$

$[2] = \{\ldots, -2, 0, 2, \ldots\}$

Since $\mathbb{Z} = [1] \cup [2]$, we must have found all equivalence classes.