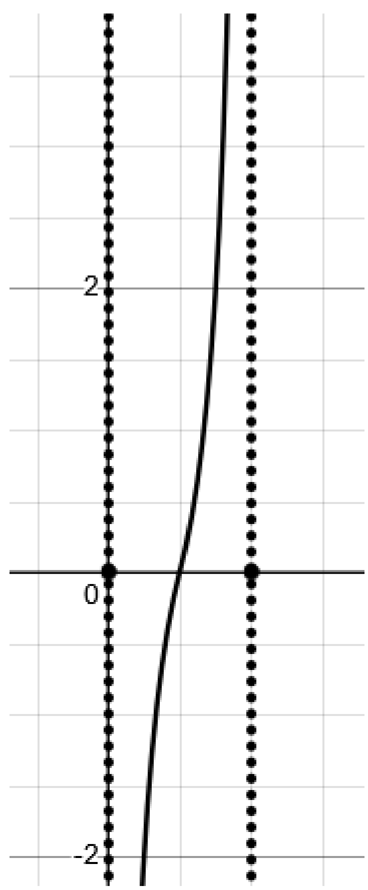
Exercise 5.4.18 Find an example of a closed bounded interval $[a, b]$ and a function $f\colon [a, b] \to \mathbb{R}$ such that f attains neither a maximum nor a minimum value on $[a, b]$.

Solution: Let $[a, b] = [0, 1]$ and define

$$f(x) = \begin{cases} -\dfrac{1}{x}, & 0 < x < \dfrac{1}{2} \\ \dfrac{1}{1-x}, & \dfrac{1}{2} \leq x < 1 \\ 0, & x = 0, 1. \end{cases},$$

pictured here:



This function does not attain a maximum or minimum on $[0, 1]$.

Example 5.4.19 Find an example of a bounded interval I and a function $f\colon I \to \mathbb{R}$ which is continuous on I such that f attains neither a maximum

or minimum value on I.
Solution: Let $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and define $f\colon I \to \mathbb{R}$ by $f(x) = \tan(x)$. This function has asymptotes at the two endpoints, goes to $+\infty$ near the right endpoint and goes toward $-\infty$ at the left endpoint. So it does not attain a max or min on I.

<u>Exercise 5.4.24</u> Suppose $D \subset \mathbb{R}$ and $f\colon D \to \mathbb{R}$ is uniformly continuous. Show that if $\{x_n\}_{n\in I}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is a Cauchy sequence in $f(D)$.
Solution: Let $\{x_n\}$ be a Cauchy sequence in D, i.e. for every $\epsilon > 0$ there exists N so that for all $m, n > N$, $|x_n - x_m| < \epsilon$. Since f is uniformly continuous, for every $\epsilon > 0$ there exists $\delta > 0$ so that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $\epsilon > 0$ and choose $\delta > 0$ so that for all $x, y \in D$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Choose $N > 0$ so that if $m, n > N$, $|x_n - x_m| < \delta$. Since $|x_n - x_m| < \delta$, we see that $|f(x_n) - f(x_m)| < \epsilon$, completing the proof that $\{f(x_n)\}$ is a Cauchy sequence. ■